Coloring the Plane with Rainbow Squares

Mike Krebs

Joint work with Richard Katz and Anthony Shaheen
The Hadwiger-Nelson problem
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Assign every point in the plane a color so that no two points of distance 1 from each other have the same color.
The Hadwiger-Nelson problem

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What is the smallest number of colors needed?
The Hadwiger-Nelson problem

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The first thing to notice is, at least three colors are needed.
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At least 4, and at most 7.
A variation on the Hadwiger-Nelson problem
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Assign every point in the plane a color.
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Assign every point in the plane a color.

Call a square a \textit{rainbow square} if its vertices all have different colors.
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Assign every point in the plane a color.

Call a square a *rainbow square* if its vertices all have different colors.

What is the smallest number of colors needed so that every unit square is a rainbow square?
A Putnam problem
A Putnam problem

A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has the property that

$$f(A) + f(B) + f(C) + f(D) = 0$$

whenever $A$, $B$, $C$, and $D$ are the vertices of a square.
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Must $f$ be the zero function?
A Putnam problem

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Must $f$ be the zero function?

Answer: Yes.
A Putnam problem

Proof:
A Putnam problem

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A Putnam problem

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\[ f(A) + f(B) + f(P) + f(D) = 0 \]
\[ f(B) + f(C) + f(E) + f(P) = 0 \]
\[ f(P) + f(E) + f(H) + f(G) = 0 \]
\[ f(D) + f(P) + f(G) + f(F) = 0 \]
A Putnam problem

Proof:

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\[ f(D) + f(P) + f(G) + f(F) = 0 \]

\[ f(A) + f(C) + f(H) + f(F) + 2f(B) + 2f(E) + 2f(G) + 2f(D) + 4f(P) = 0 \]
A Putnam problem

Proof:

\[ f(A) + f(B) + f(P) + f(D) = 0 \]
\[ f(B) + f(C) + f(E) + f(P) = 0 \]
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\[ f(P) = 0 \]
Q.E.D.
A variation on a Putnam problem
A variation on a Putnam problem

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that

$$f(A) + f(B) + f(C) + f(D) = 0$$

whenever $A$, $B$, $C$, and $D$ are the vertices of a unit square.

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Must $f$ be the zero function?

Answer: Yes.
A variation on a Putnam problem

Proof:
A variation on a Putnam problem

Proof:
Complicated.
A variation on a Putnam problem

Proof:

Complicated.
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A variation on the Hadwiger-Nelson problem

Theorem: At least 5 colors are needed.
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A variation on the Hadwiger-Nelson problem

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Proof: Temporarily assume 4 colors suffice:
A variation on the Hadwiger-Nelson problem

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Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = 3$ if $x$ is colored red, and $f(x) = -1$ otherwise.
A variation on the Hadwiger-Nelson problem

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Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = 3$ if $x$ is colored red, and $f(x) = -1$ otherwise.

Then $f$ sums to zero on the corners of every unit square.
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Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x) = 3$ if $x$ is colored red, and $f(x) = -1$ otherwise.

Then $f$ sums to zero on the corners of every unit square. So $f$ is the zero function. Contradiction. Q.E.D.
A variation on the Hadwiger-Nelson problem

Theorem: At most 13 colors are needed.

Proof:
| Other variations |
Other variations

Shapes other than squares (arbitrary finite sets)
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C. de Groote and M. Duerinckx,

*Functions with constant mean on similar countable subsets of \( \mathbb{R}^2 \),*

Other variations

Shapes other than squares (arbitrary finite sets)

Other geometric spaces (sphere, hyperbolic space)
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Higher dimensions
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Sorry if I stole your question!