Fractional Chromatic Number of Distance Graphs and Density of Integral Sets with Missing Differences

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September 9, 2001

Abstract

Let $Z$ be the set of all integers and $M$ a set of positive integers. The distance graph $G(Z,M)$ generated by $M$ is the graph with vertex set $Z$ and in which $i$ and $j$ are adjacent whenever $|i - j| \in M$.

*Supported in part by the National Science Foundation under grant DMS 9805945.
†Supported in part by the National Science Council, R. O. C., under grant NSC89-2115-M-110-012.
This paper investigates the fractional chromatic number of the distance graphs $G(Z, M)$ with clique size at least $|M|$. We first give a complete characterization of such graphs. Then determine their fractional chromatic numbers, except for one case. For the exceptional case, the upper bound and the lower bound for the fractional chromatic number presented in this article are sharp enough to determine the chromatic number for such graphs. Consequences of our results include the confirmation of a conjecture of Rabinowitz and Proulx [19] concerning the density of integral sets with missing differences.

1991 Mathematics Subject Classification: Primary 05C15 and 11B05.

Keywords: distance graphs, independence number, circular chromatic number, fractional chromatic number, $T$-coloring, integer sequences, density.

1 Introduction

Let $X$ be a metric space and $M$ a set of positive numbers. The distance graph $G(X, M)$ on $X$ generated by $M$ has vertex set $X$ and in which two points $x, y \in X$ are adjacent whenever $d(x, y) \in M$, where $d(x, y)$ is the distance between $x$ and $y$. The chromatic number of distance graphs has been studied extensively in the literature. The most well-known open problem in this area is the “plane chromatic number”: What is the least number of colors needed to color the plane so that any two points of unit distance receive distinct colors? (That is, what is the exact value of $\chi(G(R^2, \{1\}))$?) It is known that $4 \leq \chi(G(R^2, \{1\})) \leq 7$ [17], however, no substantial progress has been made in the past three decades.

Eggleton, Erdős and D. K. Skilton [10] initiated the study of the distance graphs $G(Z, M)$, which has all integers $Z$ as vertex set, and two vertices $x$ and $y$ are adjacent if and only if $|x - y| \in M$. Since then, this family of distance graphs (also called integral distance graphs) has attracted considerable attention. Most of the efforts in the past are devoted to the chromatic number of such distance graphs ([8, 10, 11, 12, 13, 16, 21, 22]).

In this paper, we are mainly focused on the fractional chromatic number of distance graphs $G(Z, M)$. A fractional coloring of a graph $G$ is a mapping
function $f$ which assigns a nonnegative weight $f(I)$ to each independent set $I$ of $G$ in such a way that for each vertex $x$, $\sum_{x \in I} f(I) \geq 1$. The fractional chromatic number $\chi_f(G)$ of $G$ is the least total weight of a fractional coloring of $G$.

The fractional chromatic number of distance graphs $G(Z, M)$ has its origination from an earlier number theory problem (called the density of integral sets with missing differences), although it was not introduced explicitly then. For a set $M$ of positive integers, a set $S$ of non-negative integers is called an $M$-set if $a - b \notin M$ for any $a, b \in S$. For any integer set $S$, denote by $S(n)$ the number of elements of the set $\{0, 1, \cdots, n\} \cap S$. The upper density $\overline{\delta}(S)$ and the lower density $\underline{\delta}(S)$ of $S$ are defined, respectively, by

$$
\overline{\delta}(S) = \lim_{n \to \infty} S(n)/n, \quad \underline{\delta}(S) = \lim_{n \to \infty} S(n)/n.
$$

We say $S$ has density $\delta(S)$ if $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The problem of interest is to maximum density $\mu(M)$ of an $M$-set, defined by

$$
\mu(M) = \sup \{ \delta(S) : S \text{ is an } M\text{-set} \}.
$$

The problem of determining or estimating $\mu(M)$ was initially posed by Motzkin in an unpublished problem collection (cf. [2]). In 1973, Cantor and Gordon [2] proved the existence of $\mu(M)$ for any $M$. In addition to [2], the values of $\mu(M)$ for several special families of $M$ have also been studied by Haralambis [15] and by Rabinowitz and Proulx [19].

It turns out that the fractional chromatic number of distance graphs and the density of $M$-sets are indeed the same problem, due to the following result proved in [4]:

**Theorem 1** For any finite set $M$, $\mu(M) = 1/\chi_f(G(Z, M))$.

Thus, throughout this article, we shall mainly use the parameter $\mu(M)$ instead of $\chi_f(G(Z, M))$, for two reasons: 1) The origination of this problem; 2) some earlier results used in some proofs of this article are formulated in terms of $\mu(M)$.

It is easy to see that $\mu(M) = \mu(aM)$ for any integer $a$, where $aM$ is the set obtained by multiplying every element in $M$ by $a$. Thus, throughout the paper, we shall assume that $\gcd(M) = 1$.  

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If $M$ has only one element, it is trivial that $\mu(M) = 1/2$. If $M = \{a, b\}$ and $a$ and $b$ are both odd, then $\mu(M) = 1/2$ (since all even numbers form an $M$-set). Cantor and Gordon [2] proved the following result which settles the case that $a, b$ are of different parities.

**Theorem 2 [2]** If $M = \{a, b\}$ where $a$ and $b$ are of different parities, then

$$\mu(M) = \frac{\lfloor (a + b)/2 \rfloor}{a + b}.$$

For $|M| \geq 3$, the values of $\mu(M)$ are known only for very limited families of $M$. In [15, 19], the cases $M = \{1, a, b\}$, $M = \{1, 2, a, b\}$ and $M = \{a, b, a+b\}$, are considered, and $\mu(M)$ are determined for some special values of $a$ and $b$.

In this article, we study a special family of sets $M$ called almost difference closed. A set $M$ is called *difference closed* if the difference of any pair of elements in $M$ falls in $M$. A set $M$ is *almost difference closed* if there exists $M' \subset M$ such that $|M'| = |M| - 1$ and $|a - b| \in M$ for any $a, b \in M'$.

Let $\omega(G)$ denote the maximum clique size of $G$. In the terminology of graph theory, the study of $\mu(M)$ for almost difference closed sets $M$ is the same as the study of $\chi_f(G(Z, M))$ for the distance graphs $G(Z, M)$ with $\omega(G(Z, M)) \geq |M|$. It is easy to verify that $\omega(G(Z, M)) \geq t$ if and only if there is a subset $M' \subset M$, $|M'| = t - 1$, such that $|x - y| \in M$ for any $x, y \in M'$. Hence, $M$ is almost difference closed if and only if $\omega(G(Z, M)) \geq |M|$, and $M$ is difference closed if and only if $\omega(G(Z, M)) = |M| + 1$.

In Section 2, we give a characterization of almost difference closed sets by categorizing them into three types. The values of $\mu(M)$ for these three types are studied in Sections 3, 4 and 5, respectively. The exact values of $\mu(M)$ for most of the almost difference closed sets $M$ are determined. The only exception is a sub case within the third type, namely,

$$M = \{x, y, y - x, x + y\}, \ x \text{ and } y \text{ are both odd, } 1 < x < y.$$  

For this exceptional case, we prove a lower bound and an upper bound for $\mu(M)$. 

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Note if $|M| \leq 2$ or $M = \{a, b, a+b\}$, then $M$ is almost difference closed. Thus, results in this article are generalizations of some previously known results. In particular, a special case of our results settles in affirmative a conjecture on the value of $\mu(M)$ for $M = \{a, b, a+b\}$, proposed by Rabinowitz and Proulx [19].

Our work reveals an interesting property shared by most of the almost difference closed sets $M$. For any integers $n$ and $m$, denote $\phi_m(n) = \left\lfloor \frac{n}{m} \right\rfloor$. Let $|M| = m$, and set

$$\beta(M) = \max\{\phi_m(a+b)/(a+b) : a, b \in M, a \neq b, m \not\mid (a+b)\}. \quad (1.2)$$

We shall prove that $\mu(M)$ is equal to either $1/m$ or $\beta(M)$, for almost difference closed sets $M$ which are not in the form of (1.1). Moreover, in most cases, $\mu(M) = \kappa(M)$, where $\kappa(M)$ is a parameter related to Diophantine approximations, as well as to view obstruction problems in geometry [5, 6, 7, 9, 23, 24, 25, 26, 27]. If $M$ is in the family of (1.1), then the lower bound and the upper bound for $\mu(M)$ we obtain indicate that $\mu(M)$ is not equal to any of the following: $\kappa(M)$, $1/m$, $\phi_m(a+b)/(a+b)$ for any $a, b \in M$. In Sections 2 and 6, we give further discussion on the relations between $\mu(M)$ and $\kappa(M)$, and use those relations to calculate the chromatic number and the circular chromatic number of the distance graphs generated by almost difference closed sets.

## 2 Characterization of almost difference closed sets

It is well-known that for any graph $G$, $\omega(G) \leq \chi_f(G) \leq \chi(G)$. So

$$1/\chi(G(Z, M)) \leq \mu(M) \leq 1/\omega(G(Z, M)). \quad (2.1)$$

Denote $|M| = m$. As observed in Section 1, $M$ is almost difference closed if and only if $\omega(G(Z, M)) \geq m$. On the other hand, it is known [8, 21] that for any $M$, $\chi(G(Z, M)) \leq m + 1$. Therefore, for almost difference closed set $M$, we have

$$1/(m+1) \leq \mu(M) \leq 1/m.$$
Many families of sets $M$ for which the values of $\mu(M)$ have been investigated in earlier literature are special cases of almost difference closed sets.

The following is a characterization of difference closed sets and almost difference closed sets.

**Theorem 3** Suppose $M$ is a finite set of positive integers with $|M| = m$ and $\gcd M = 1$. Then $M$ is difference closed if and only if $M = \{1, 2, \ldots, m\}$ and $M$ is almost difference closed if and only if $M$ is one of the following three types:

A.1) $M = \{a, 2a, 3a, \ldots, (m - 1)a, b\}$.

A.2) $M = \{a, b, a + b\}$ for some $b \neq 2a$.

A.3) $M = \{x, y, y - x, y + x\}$ for some $y \neq 2x$.

**Proof.** It is obvious that $M$ is difference closed if and only if $M = \{1, 2, \ldots, m\}$. It is also easy to verify that each of the sets $M$ listed in A.1, A.2 and A.3 is almost difference closed. Indeed, $\{a, 2a, \ldots, (m - 1)a\}$, $\{a, a + b\}$, $\{x, y, x + y\}$ can be used as the required set $M'$ for Types A.1, A.2 and A.3, respectively.

Suppose $M$ is an almost difference closed set and $|M| = m$. If $m \leq 2$, then every set $M$ is of Type A.1.

Suppose $m = 3$ and $M' = \{x, z\}$ ($x < z$) is a required subset of $M$. So $z - x \in M$. If $z - x = x$, then $M$ is of Type A.1. Otherwise $M = \{x, z - x, z\}$ is of Type A.2.

Suppose $m = 4$ and $M' = \{x, y, z\}$ ($x < y < z$) is a required subset of $M$. Let $M - M' = \{t\}$. As $y - x \in M$ and $y - x \notin \{y, z\}$, we conclude that $y - x \notin \{x, t\}$. If $y - x = t$, then $z - x \notin \{x, y\}$. If $z - x = x$, then $z - y = z - x - t = x - t \notin M$, a contradiction. If $z - x = y$, then $M$ is of Type A.3.

Thus, assume $y - x = x$. Then $y = 2x$ and $\{z - y, z - x\} \subset \{x, 2x, t\}$. Since $z - x = z - y + y - x = z - y + x$, we conclude that either $z - y = x$, or $z - y = 2x$ and $t = 3x$. Any of these two cases implies that $M$ is of Type A.1.
Suppose $m \geq 5$. Assume that $M$ is not of Type A.1. Let $M' = \{x_1, x_2, \ldots, x_{m-1}\}$ be a required subset of $M$, where $x_1 < x_2 < \cdots < x_{m-1}$. Let $M - M' = \{t\}$. Set

$$i_0 := \min\{j : x_j \neq jx_1\}.$$ 

Then $2 \leq i_0 \leq m - 1$ (because $M$ is not of Type A.1.).

**Claim:** $x_1 \neq 2t$ and $t \neq i_0x_1$.

**Proof of Claim:** We first prove that $x_1 \neq 2t$. Suppose $x_1 = 2t$. Then $x_2 - x_1 \in \{t, 2t\}$, so $x_2 \in \{3t, 4t\}$. If $x_2 = 4t$, then $\{x_3 - x_2, x_3 - x_1\} \subseteq \{t, 2t, 4t\}$. This implies that $x_3 = 6t$. Repeatedly applying the same argument, we get $x_j = 2jt$ for all $1 \leq j \leq m - 1$, implying $M$ is of Type A.1.

Thus, assume $x_2 = 3t$. Set

$$j_0 := \min\{j : x_j \neq (j + 1)t\}.$$ 

As $M$ is not of Type A.1, $j_0$ exists and $3 \leq j_0 \leq m - 2$. Let

$$z = x_{j_0} - x_{j_0 - 1} \neq t.$$ 

Then one has

$$x_{j_0} - x_j = z + (j_0 - j - 1)t, \text{ for all } 1 \leq j \leq j_0 - 2.$$ 

Because all the differences above must fall in $M$ and are smaller than $x_{j_0}$, we have

$$\{z, z + t, z + 2t, \ldots, z + (j_0 - 2)t\} \subseteq \{t, x_1, x_2, \cdots, x_{j_0-1}\} = \{t, 2t, 3t, \ldots, j_0t\}.$$ 

Thus we conclude that $z = 2t$ and $x_{j_0} = (j_0 + 2)t$.

Let $w = x_{j_0+1} - x_{j_0}$. Since $j_0 \leq m - 2$, $x_{j_0+1}$ exists. And $x_{j_0+1} - x_j = x_{j_0+1} - x_{j_0} + x_{j_0} - x_j = w + x_{j_0} - x_j$, for all $1 \leq j \leq j_0 - 1 \in M$, so

$$\{w, w + 2t, w + 3t, \ldots, w + j_0t\} \subseteq \{t, x_1, x_2, \cdots, x_{j_0-1}, x_{j_0}\} = \{t, 2t, 3t, \ldots, j_0t, (j_0 + 2)t\}.$$
This is impossible. Since if \( w = t \), then \( w + j_0t \notin \{ t, 2t, 3t, \ldots, j_0t, (j_0 + 2)t \} \); if \( w = 2t \), then \( w + (j_0 - 1)t \notin \{ t, 2t, 3t, \ldots, j_0t, (j_0 + 2)t \} \). Therefore, we conclude that \( x_1 \neq 2t \).

To prove \( t \neq i_0x_1 \), we use a similar approach. Assume \( t = i_0x_1 \). Then \( \{ t, x_1, x_2, \ldots, x_{i_0-1} \} = \{ x_1, 2x_1, \ldots, i_0x_1 \} \). Let

\[
 j_0 := \min\{ j : x_j \neq (j + 1)x_1 \},
\]

Then \( i_0 \leq j_0 \leq m - 2 \), as \( M \) is not of Type A.1. Similar argument as in the above derives a contradiction. This completes the proof of the Claim. \( \square \)

Since \( x_{i_0} - x_{i_0-1} \in M \), \( x_{i_0} - x_{i_0-1} \neq x_1 \) (by the choice of \( i_0 \)) and \( x_{i_0} - x_{i_0-1} < x_j \) for \( j \geq i_0 \), we conclude that either \( x_{i_0} - x_{i_0-1} = t \) or \( x_{i_0} - x_{i_0-1} = x_j \) for some \( 2 \leq j \leq i_0 - 1 \).

**Case 1**: \( x_{i_0} - x_{i_0-1} = t \)

If \( i_0 \geq 3 \), then \( x_{i_0} - x_{i_0-2} = (x_{i_0} - x_{i_0-1}) + (x_{i_0-1} - x_{i_0-2}) = t + x_1 < x_{i_0} \in M \). This implies that \( x_1 + t = kx_1 \) for some \( k < i_0 \), so \( t = (k - 1)x_1 = x_{k-1} \), a contradiction.

Thus assume \( i_0 = 2 \). Because \( x_3 - x_1 = (x_3 - x_2) + t \), we have

\[
 (x_3 - x_2), (x_3 - x_2) + t \in \{ t, x_1, x_1 + t \}
\]

This implies that \( x_3 - x_2 \in \{ t, x_1 \} \). If \( x_3 - x_2 = t \), then \( x_1 = 2t \), contrary to the Claim.

Hence, assume \( x_3 - x_2 = x_1 \). Let \( z = x_4 - x_3 \). Then

\[
 \{ z, z + x_1, z + x_1 + t \} \subseteq \{ t, x_1, x_2, x_3 \} \\
= \{ t, x_1, x_1 + t, 2x_1 + t \}.
\]

This implies that \( z = x_1 \) and \( t = 2x_1 \), which is a contradiction by the Claim.

**Case 2**: \( x_{i_0} - x_{i_0-1} = x_j = jx_1 \) for some \( 2 \leq j \leq i_0 - 1 \)

Then \( i_0 \geq 3 \), and

\[
 x_{i_0} - x_{j-1} = (x_{i_0} - x_{i_0-1}) + (x_{i_0-1} - x_{j-1}) \\
= jx_1 + (i_0 - 1 - j + 1)x_1 \\
= i_0x_1 \in M.
\]
This implies that $t = i_0 x_1$, contrary to the Claim.

In the next three sections, we shall study the values of $\mu(M)$ for all the almost difference closed sets $M$, by considering these three types separately.

A useful tool for establishing a lower bound for $\mu(M)$ is to calculate the parameter $\kappa(M)$. For a real number $x$, let $\{x\}$ be the fractional part of $x$, i.e., $\{x\} = x - \lfloor x \rfloor$, and let $||x||$ denote the distance from $x$ to the nearest integer, i.e., $||x|| = \min\{\{x\}, 1 - \{x\}\}$. For a set $X$ of real numbers, and $t$ a real number, let $||tX|| = \inf\{||tx|| : x \in X\}$, and let

$$\kappa(X) = \sup\{||tX|| : t \in R\}.$$ 

It was proved in [2] that for any set $M$, it is always true that

$$\mu(M) \geq \kappa(M). \quad (2.2)$$

Thus we use $\kappa(M)$ as a lower bound for $\mu(M)$.

To establish upper bounds for $\mu(M)$, we shall use the following lemma proved in [15]:

**Lemma 4** [15] Let $M$ be a set of positive integers, $\alpha$ a real number in the interval $(0,1]$. If $\mu(M) \geq \alpha$ then there is an $M$-set $S$ such that for any $n \geq 0$, $S(n) \geq \alpha(n + 1)$, where $S(n) = |S \cap \{0, 1, \cdots , n\}|$. In particular, $S(0) \geq \alpha$ implies that $0 \in S$.

Another tool for establishing upper bounds for $\mu(M)$ is to consider the fractional chromatic number of subgraphs of $G(Z, M)$. We shall denote by $G[n]$ the subgraph of $G(Z, M)$ induced by the set $\{0, 1, \cdots , n\}$ (so $G[n]$ has $n + 1$ vertices). Then $\chi_f(G(Z, M)) \geq \chi_f(G[n])$, and hence

$$\mu(M) = 1/\chi_f(G(Z, M)) \leq 1/\chi_f(G[n]), \text{ for any } n \geq 0.$$ 

This tool is not explicitly used in our proof, however, some proofs are based on ideas from fractional colorings of $G[n]$. 

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3 Almost difference closed sets: Type A.1

This type turns out to be the easiest. We settle both the values of $\mu(M)$ and $\kappa(M)$, which are always equal for this case. For a set $A$ and an integer $a$, let $A + a = \{x + a : x \in A\}$.

**Theorem 5** Suppose $M = \{a, 2a, \ldots, (m - 1)a, b\}$, where $\gcd(a, b) = 1$. If $a = 1$, then

$$\mu(M) = \kappa(M) = \begin{cases} \frac{1}{m}, & \text{if } b \text{ is not a multiple of } m; \\ \frac{k}{km+1}, & \text{if } b = km \text{ for some } k. \end{cases}$$

If $a \geq 2$, then $\mu(M) = \kappa(M) = 1/m$.

**Proof.** Assume $a = 1$. If $b$ is not a multiple of $m$, let $t = 1/m$. We then have $||tM|| = 1/m$, so $\mu(M) \geq \kappa(M) \geq 1/m$. Hence the equality holds by (2.2).

Now assume that $M = \{1, 2, 3, \ldots, (m - 1), mk\}$ for some integer $k \geq 1$. Let $t = k/(mk + 1)$. Then $||Mt|| = k/(mk + 1)$, so $\mu(M) \geq \kappa(M) \geq k/(mk + 1)$. To prove that $\mu(M) \leq k/(mk + 1)$, by Lemma 4, it suffices to show that for any $M$-set $S$ with $0 \in S$, we have $S(mk) \leq k$. As $0 \in S$, one has $b = mk \notin S$.

Partition the set $\{0, 1, 2, \ldots, mk - 1\}$ into

$$X_i = \{0, 1, \ldots, m - 1\} + im, \text{ for } i = 0, 1, \ldots, k - 1.$$ 

Obviously $|S \cap X_i| \leq 1$, so $S(mk) \leq k$. Therefore $\mu(M) \leq k/(mk + 1)$, which implies that $\mu(M) = \kappa(M) = k/(km + 1)$.

Assume that $a \geq 2$. We shall prove that $\mu(M) = \kappa(M) = 1/m$. By (2.2), it suffices to prove that $\kappa(M) \geq 1/m$, i.e., there is a real number $t$ such that $||Mt|| \geq 1/m$. Assume $b = q(am) + r$ for some integer $q$ and $0 \leq r < am$. Then $r > 0$, since $\gcd(a, b) = 1$. If $a \leq r \leq (m - 1)a$, then let $t = 1/(am)$. It is easy to verify that $||Mt|| = 1/m$.

Now assume that $1 \leq r \leq a - 1$. Let $l$ be the smallest integer such that $rl \geq a$. Then $2 \leq l \leq a$, and $a \leq rl \leq 2a - 1$. Furthermore, there exists some integer $k \in \{0, 1, 2, \ldots, m - 3\}$ such that $l + k \equiv 1$ or $-1 \pmod{m}$, so $a(l + k) \equiv a$ or $-a \pmod{am}$.
Take \( t = (l + k)/(am) \). Then \( ||bt|| = ||r(l + k)/(am)|| \geq 1/m \), since

\[
\begin{align*}
a &\leq lr \\
&\leq (l + k)r \\
&\leq (l + m - 3)r \\
&= (l - 1)r + (m - 2)r \\
&< a + (m - 2)a \\
&= (m - 1)a.
\end{align*}
\]

Because \( a(l + k) \equiv a \) or \(-a \) (mod \( am \)), one can easily verify that \( ||t(ja)/(am)|| \geq 1/m \) for \( j = 1, 2, \cdots, m - 1 \). Hence we conclude that

\[ \kappa(M) \geq ||tM|| \geq 1/m. \]

The case for \((m - 1)a + 1 \leq r \leq am - 1\) is symmetric, and we omit the details.

\[ \blacksquare \]

### 4 Almost difference closed sets: Type A.2

The parameter \( \mu(M) \) is also known to be the same as the asymptotic \( T \)-coloring ratio. Let \( T \) be a set of non-negative integers with \( 0 \in T \). A \( T \)-coloring on a graph \( G \), with span \( k \), is a function \( f : V(G) \to \{0, 1, 2, 3, \cdots, k\} \) such that \( |f(u) - f(v)| \not\in T \) if \( u \sim v \). For a given set \( T \), denote the minimum span of \( K_n \) by \( \sigma_n(T) \). The asymptotic \( T \)-coloring ratio is defined as

\[ rt(T) = \lim_{n \to \infty} \sigma_n(T)/n. \]

The asymptotic \( T \)-coloring ratio has been studied by Rabinowitz and Proulx [19] and by Griggs and Liu [14]. It was noted in [14] that \( \mu(M) = rt(T) \), if \( T = M \cup \{0\} \). In this section, some known results quoted in terms of \( \mu(M) \) are indeed originally given in the form of \( rt(T) \). Due to the equivalence of these two parameters, we shall consistently use \( \mu(M) \).

Suppose \( M = \{a, b, a + b\} \). If none of \( a, b \) or \( a + b \) is a multiple of 3, then it is easy to prove that \( \kappa(M) = \mu(M) = 1/3 \) ([19]) (all multiples of 3 form an \( M \)-set). If \( a = 1 \), i.e. \( M = \{1, b, b + 1\} \), then the value of \( \mu(M) \) was determined in [15].
For other sets $M$ of the form $M = \{a, b, a + b\}$, a lower bound for $\mu(M)$ was proved in [19] and the authors conjectured that the lower bound is sharp:

**Conjecture 1** [19] Suppose $M = \{a, b, a + b\}$, $\gcd(a, b) = 1$, and one of $a, b, a + b$ is a multiple of 3. Then

$$\mu(M) = \max\{\frac{(2b + a)/3}{2b + a}, \frac{(2a + b)/3}{2a + b}\}.$$

In this section, we prove this conjecture, and hence settle the values of $\mu(M)$ for all the sets $M$ of Type A.2.

It was proved in [19] that under the assumption of Conjecture 1,

$$\mu(M) \geq \max\{\frac{(2b + a)/3}{2b + a}, \frac{(2a + b)/3}{2a + b}\}.$$

Thus to prove Conjecture 1, it suffices to prove that

$$\mu(M) \leq \max\{\frac{(2b + a)/3}{2b + a}, \frac{(2a + b)/3}{2a + b}\}.$$

By discussing different cases, the inequality above is implied by the following theorem.

**Theorem 6** Suppose $0 < a < b$ are integers with $\gcd(a, b) = 1$. Let $c = a + b$ and $M = \{a, b, c\}$.

- If $b - a = 3k$, then $\mu(M) \leq 1/3$.
- If $b - a = 3k + 1$, then $\mu(M) \leq (a + k)/(3a + 3k + 1)$.
- If $b - a = 3k + 2$, then $\mu(M) \leq (a + 2k + 1)/(3a + 6k + 4)$.

**Proof.** The case for $b - a = 3k$ in Theorem 6 has been proved in [19].

**Case 1:** $b - a = 3k + 1$
Suppose $S$ is an $M$-set with $0 \in S$. By Lemma 4, it suffices to show that there exists an integer $n$ satisfying the inequality

$$S(n) \leq \frac{a + k}{3a + 3k + 1}(n + 1) \quad (4.1)$$

First, consider $S(c + a - 1)$. Since $0 \in S$, we know that $a, b, c \notin S$. For $i = 1, 2, \cdots, a - 1$, define the Triangle-$i$ by

$$T_i = \{i, i + b, i + c\}.$$ 

Since the difference of any two elements of $T_i$ lies in $M$, one has $|S \cap T_i| \leq 1$. Set

$$I = \{1 \leq i \leq a - 1 : S \cap T_i = \emptyset\},$$

$$T = S \cap \{a + 1, a + 2, \cdots, b - 1\}, \text{ and } t = |T|.$$ 

Hence $S(c + a - 1) = t + a - |I|$. If $t \leq k$, then $S(c + a - 1) \leq a + k$. Since $c + a - 1 = 3a + 3k$, (4.1) is satisfied by letting $n = c + a - 1$.

Now assume that $t \geq k + 1$. We shall prove that (4.1) holds when $n = b + c - 1$. Note that $b + c - 1 = 3a + 6k + 1$. Since $t \geq k + 1$, it can be verified that

$$\frac{a + k}{3a + 3k + 1}(3a + 6k + 2) \geq a + 2k \geq a + 3k + 1 - t.$$ 

Hence, to show that (4.1) holds for $n = b + c - 1$, it suffices to show that

$$S(b + c - 1) \leq a + 3k + 1 - t. \quad (4.2)$$

Observe that $S(b + c - 1) = S(c + a - 1) + |S \cap \{c + a, c + a + 1, \cdots, c + b - 1\}|$.

Let

$$U = \{c + a, c + a + 1, \cdots, c + b - 1\} - S.$$ 

Then

$$S(b + c - 1) = a + t - |I| + (b - a) - |U|.$$ 

Note that $a + 3k + 1 - t = a + t + (b - a) - 2t$. Hence, to prove (4.2), it amounts to show that

$$|I| + |U| \geq 2t.$$ 

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Thus it suffices to prove that, for each $i \in T$, we can find two distinct elements, $i_1, i_2 \in I \cup U$, such that \{i_1, i_2\} and \{j_1, j_2\} are disjoint whenever $i \neq j$.

For any $i \in T$, let $i_1 = i + c$, then $i_1 \in U$. The element $i_2$ is defined by the following two cases:

1) If $i \geq 2a$, let $i_2 = i + b$, then $i_2 \in U$. It is obvious that $i_1 \neq i_2$.

2) If $i < 2a$, then $1 \leq i - a \leq a - 1$. Consider the triangle

\[ T_{i-a} = \{i - a, i - a + b, i - a + c\}. \]

Since $i \in S$ and both the differences of $i$ and $i - a$ and of $i$ and $i - a + c$ lie in $M$, we conclude that exactly one of the following is true:

\[ S \cap T_{i-a} = \emptyset \quad \text{or} \quad S \cap T_{i-a} = \{i - a + b\}. \]

If $S \cap T_{i-a} = \emptyset$, let $i_2 = i - a$, then $i_2 \in I$ and $i_1 \neq i_2$.

If $S \cap T_{i-a} = \{i - a + b\}$, let $i_2 = i + 2b - a$, then $i_2 \in U$ (because $i + b - a \in S$ and $(i + 2b - a) - (i + b - a) = b \in M$). Since $b \neq 2a$ (for otherwise the difference of $i$ and $i + b - a$ lies in $M$), one has $i_1 \neq i_2$.

Now it remains to show that for $i, j \in T$, if $i \neq j$ then the sets \{i_1, i_2\} and \{j_1, j_2\} are disjoint. Without loss of generality, we may assume that $i < j$. It is obvious that $i_1 \neq j_1$.

If $i_2 = j_2$, then since $i < j$, we only have to check the case that $i_2 = i + 2b - a = j_2 = j + b$. However, this would imply that $j = i + b - a > b$ (since $i \geq a + 1$), a contradiction (recall that $T = \{a + 1, \ldots, b - 1\} \cap S$).

If $i_2 = j_1 = j + c$, then we must have $i_2 = i + 2b - a$ (since $i < j$). Hence $j = i + b - 2a$. By definition of $i_2$, we have $i + b - a \in S$. This is impossible, because then the difference of $j$ and $i + b - a$ (both are elements of $S$) is $a$, which lies in $M$.

If $j_2 = i_1 = i + c$, then either $j_2 = j + b$ or $j_2 = j + 2b - a$. The former case implies that $j - i = a$, a contradiction, since $j, i \in S$. By definition of $j_2$, the latter case implies that $j + b - a \in S$. This is again a contradiction,
since the difference of \( j + b - a \) and \( i \) (both are elements of \( S \)) is \( a \), which lies in \( M \).

Therefore we conclude that the two sets \( \{i_1, i_2\} \) and \( \{j_1, j_2\} \) are disjoint. This completes the proof of Case 1.

**Case 2: \( b - a = 3k + 2 \)**

The proof of this case is similar to the one of Case 1. Suppose \( S \) is an \( M \)-set with \( 0 \in S \). By Lemma 4, we shall prove that there exists an integer \( n \) satisfying the inequality

\[
S(n) \leq \frac{a + 2k + 1}{3a + 6k + 4}(n + 1).
\]

First, we consider \( S(c + a - 1) \). Similar to Case 1, set

\[
T_i = \{i, i + b, i + c\}, \text{ for } i = 1, 2, \ldots, a - 1,
\]

\[
I = \{1 \leq i \leq a - 1 : S \cap T_i = \emptyset\},
\]

\[
T = S \cap \{a + 1, a + 2, \ldots, b - 1\}, \text{ and } t = |T|.
\]

Hence \( S(c + a - 1) = t + a - |I| \). If \( t \leq k \), then \( S(c + a - 1) \leq a + k \). This implies that (4.3) holds when \( n = c + a - 1 = 3k + 3a + 1 \), because

\[
a + k \leq \frac{a + 2k + 1}{3a + 6k + 4}(3a + 3k + 2).
\]

Next, assume \( t \geq k + 1 \). By the same argument used in Case 1 (the one about assigning \( i_1 \) and \( i_2 \) to each \( i \in T \)), one can show that

\[
S(b + c - 1) = S(3a + 6k + 3) \leq a + t + (b - a) - 2t \leq a + 2k + 1.
\]

Hence (4.3) is satisfied when \( n = b + c - 1 = 3a + 6k + 3 \). This completes the proof of Theorem 6.

**5 Almost difference closed sets: Type A.3**

This type, \( M = \{x, y, y - x, x + y\}, \ y > x \), turns out to be the most complicated family of almost difference closed sets, in terms of the values of \( \mu(M) \).
Although a special case when \( x \) and \( y \) are of different parity is rather simple, difficulty emerges when \( x \) and \( y \) are both odd (cf. (1.1)). We will determine the values of \( \mu(M) \) for the former case and give an upper bound and a lower bound for the latter case. The lower bound reveals an interesting fact that \( \mu(M) \) for this case is very different from the ones for other almost difference closed sets.

It was proved in [16] that if \( x \) and \( y \) are of distinct parity, then \( \chi(G(Z,M)) = 4 \). Hence \( \chi(G(Z,M)) = \omega(G(Z,M)) = 4 \). So by (2.1), we have the following result.

**Theorem 7** If \( M = \{x,y,y-x,x+y\}, y > x, \) and \( x, y \) are of distinct parity, then \( \mu(M) = 1/4 \).

Because \( \gcd(a,b) = 1 \), by Theorem 7, for Type A.3, we now only have to consider the family of sets \( M \) that \( M = \{x,y,y-x,x+y\} \), where \( y > x \), and \( x, y \) are both odd.

In the remaining of this section, we assume that \( M = \{x,y,y-x,x+y\} \), where \( x = 2k + 1, y = 2m + 1, k < m, \) and \( \gcd(x,y) = 1 \). The following is a lower bound for \( \mu(M) \).

**Theorem 8** If \( M \) is a set defined in the above, then \( \mu(M) \geq \frac{(k+1)m}{4(k+1)m+1} \).

**Proof.** Let \( n = xy + y - x = 4(k+1)m + 1 \). Define a set \( S \) as follows:

\[
I = \{0, 2x, 4x, \cdots, (m-1)2x\},
\]

\[
Y = \bigcup_{i=0}^{k} \{I + 2iy \pmod{n}\},
\]

\[
S = \bigcup_{k=0}^{\infty} (Y + kn).
\]

We shall prove that \( S \) is an \( M \)-set with density \( \frac{(k+1)m}{n} \). Note that \( S \) is “periodic” with period \( n = xy + y - x \). To be precise, for any \( k \geq 1 \), \( S \cap \{kn, kn+1, \cdots, (k+1)n - 1\} = (S \cap \{0,1,\cdots,n-1\}) + kn \). Thus to show that \( S \) has density \( \frac{(k+1)m}{n} \), it suffices to show that

\[
|S \cap \{0,1,\cdots,n-1\}| = (k+1)m.
\]
To prove this, it suffices to show that for \(i \neq i', (I + 2iy) \cap (I + 2i'y) = \emptyset\). Otherwise there exist \(i \neq i'\) and \(j \neq j'\), \(0 \leq i \leq i' \leq k\), \(0 \leq j, j' \leq m - 1\), such that one of the following holds:

\[
2jx + 2iy = 2j'x + 2i'y, \quad \text{or} \quad 2jx + 2iy = 2j'x + 2i'y + xy + y - x.
\]

This implies that either \((j - j')x = (i' - i)y\) or \((2(j - j') + 1)x = 2(i' - i - 1)y + xy\). Neither of these is possible, since \(2|j - j'| < y - 1\), and \(\gcd(x, y) = 1\).

Now it remains to show that \(S\) is an \(M\)-set. Observe that for any \(u \in I + 2iy, v \in I + 2i'y\), letting

\[
u = 2tx + 2iy, v = 2t'x + 2i'y,\text{ for some } 0 \leq t, t' \leq m - 1, 0 \leq i, i' \leq k,\]

one has

\[
2x < |u - v| = |2(t - t')x + 2(i - j)y| \leq (y - 3)x + (x - 1)y = 2xy - 3x - y < 2n - (x + y).
\]

Therefore, it suffices to show that for any such \(u\) and \(v\),

\[
|u - v| \notin \{y, (y - x), (y + x), n \pm x, n \pm y, n \pm (y - x), n \pm (y + x)\}.
\]

Note that \(|u - v|\) is even, so it remains to show that

\[
|u - v| \notin \{y - x, y + x, n \pm x, n \pm y\}.
\]

By definition of \(u\) and \(v\), we have

\[
|2(t - t')| \leq 2(m - 1) = y - 3, \quad (5.1)
\]

\[
|2(i - i')| \leq 2k = x - 1. \quad (5.2)
\]

Without loss of generality, we assume that \(u - v = 2(t - t')x + 2(i - i')y > 0\).

If \(2(t - t')x + 2(i - i')y = y \pm x\), then \((2t - 2t' \pm 1)x + (2i - 2i' - 1)y = 0\), which is impossible, because \(0 < |2t - 2t' \pm 1| \leq y - 2\) (by \((5.1)\)), and \(\gcd(x, y) = 1\).

If \(2(t - t')x + 2(i - i')y = n + x = xy + y\), then \((2t - 2t')x = (2i' - 2i + x + 1)y\).

This is impossible, because \(|2i' - 2i + x + 1| > 0\) (by \((5.2)\)), \(|2t - 2t'| < y\) (by \((5.1)\)), and \(\gcd(x, y) = 1\).
If \(2(t - t')x + 2(i - i')y = n - x = xy + y - 2x\), then \(2(t - t' + 1)x = (2i' - 2i + x + 1)y\). Again this is impossible, since \(2i' - 2i + x + 1 \neq 0\) (by (5.2)), \(|2(t - t' + 1)| < y\) (by (5.1)), and \(\gcd(x, y) = 1\).

By a similar argument, one can show that \(2(t - t')x + 2(i - i')y = n - y\). This is impossible, since \(2i' - 2i + x + 1 \neq 0\) (by (5.2)), \(j_2(t - t' + 1) < y\) (by (5.1)), and \(\gcd(x, y) = 1\).

We conjecture that the equality in Theorem 8 always holds.

**Conjecture 2** If \(M = \{x, y, y - x, y + x\}\) where \(y > x\), \(x = 2k + 1\), \(y = 2m + 1\) and \(\gcd(x, y) = 1\), then \(\mu(M) = \frac{(k+1)m}{4(k+1)m+1}\).

The following result confirms the conjecture above for the case that \(x = 1\).

**Theorem 9** If \(M = \{1, 2m, 2m + 1, 2m + 2\}\) for some \(m > 1\), then \(\mu(M) = \kappa(M) = m/(4m + 1)\).

**Proof.** By Theorem 8, it suffices to show that \(\mu(M) \leq m/(4m + 1)\). Let \(S\) be an \(M\)-set. By Lemma 4, it suffices to show that if \(0 \in S\), then \(S(4m) \leq m\). Partition the set of integers \(\{0, 1, \cdots, 4m\} - \{2m\}\) into

\[R_i = \{i, i + 1, i + 2m + 1, i + 2m + 2\}, i = 0, 2, \cdots, 2m - 2.\]

Then \(|S \cap R_i| \leq 1\). Furthermore, since \(0 \in S\), we have \(2m \not\in S\), implying that \(S(4m) \leq m\).

Let \(\beta = \frac{1}{(k+1)m}\). Then Theorem 8 says that \(\mu(M) \geq \frac{1}{4+\beta}\). In the following, we establish an upper bound for \(\mu(M)\).

**Theorem 10** Let \(\delta = \frac{1}{k^2 + 2km + 3k + m + 1}\). Then \(\mu(M) \leq \frac{1}{4+\delta}\).

**Proof.** Let \(S\) be an \(M\)-set of maximum density. By some result in [14] and [2], we may assume that \(S\) is periodic, i.e., there is an integer \(n\) such that for any \(k \geq 0\), \(S \cap \{kn, kn + 1, \cdots, (k + 1)n - 1\} = B + kn\), for some subset \(B\) of \(\{0, 1, \cdots, n - 1\}\).

In the remaining of the proof, we regard \(B\) as a subset of \(Z_n\). All calculations are carried out in the group \(Z_n\) (i.e., modulo \(n\)). For instance, for each
\[ i \in \mathbb{Z}_n, \ B + i = \{b + i : b \in B\} \subseteq \mathbb{Z}_n \] (\( b + i \) is carried out in \( \mathbb{Z}_n \)). Let \( |B| = q \).

We shall frequently use the fact that \( |B + i| = q \) for any \( i \). The density of \( S \) is equal to \( q/n \). Let \( j_{B} = q \).

We shall frequently use the fact that
\[ |B + i| = q \] for any \( i \).

The density of \( S \) is equal to \( q/n \). Let \( z = n - 4q \). To complete the proof, it suffices to show that \( z \geq \delta q = \frac{q}{k^2 + 2km + 3k + m + 1} \).

Because \( S \) is an \( M \)-set, the set \( B \) satisfies the condition that for any \( i, j \in B, i - j \pmod{n} \notin M \). Equivalently, \( B + i \cap (B + j) = \emptyset \) if \( |i - j| \in M \).

**Claim 1**
\[ |(B \cup (B + y)) \cap (B + (xy + x))| \geq q - (m + 1)z. \quad (5.3) \]

**Proof of Claim 1:** Let
\[ A = B \cup (B + y) \cup (B + x) \cup (B + (x + y)) \cup (B + 2x) \cup (B + (2x + y)), \]
\[ C = (B \cup (B + y)) - ((B + 2x) \cup (B + (y + 2x))). \]

We first show that \( |C| \leq z \).

Note that
\[ B, B + y, B + x, B + (x + y) \]
are pairwise disjoint, and
\[ B + x, B + 2x, B + (x + y), B + (y + 2x) \]
are pairwise disjoint. Hence
\[ |C| = |A - ((B + x) \cup (B + (x + y)) \cup (B + 2x) \cup (B + (2x + y)))| \leq |A| - 4q \leq z. \]

Next, we show that
\[ |(B \cup (B + y)) - ((B + (xy + x)) \cup (B + (xy + x + y))| \leq (m + 1)z. \]

Observe that \( 2(m + 1)x = xy + x \). So
\[
|\left((B \cup (B + y)) - ((B + (xy + x)) \cup (B + (xy + x + y))\right)|
= \left|\left((B \cup (B + y)) - ((B + 2(m + 1)x) \cup (B + 2(m + 1)x + y))\right)\right|
\]
\[
\leq \sum_{j=0}^{m} |((B + 2jx) \cup (B + 2jx + y)) - ((B + 2(j + 1)x) \cup (B + 2(j + 1)x + y))| \\
= \sum_{j=0}^{m} |C + 2jx| \\
\leq (m + 1)z.
\]

As each of \((B \cup (B + y))\) and \(((B + (xy + x)) \cup (B + (xy + x + y))\) has cardinality \(2q\), we conclude that

\[
|(B \cup (B + y)) \cap ((B + (xy + x)) \cup (B + (xy + x + y)))| \geq 2q - (m + 1)z. \tag{5.4}
\]

This implies \((5.3)\), since \(|B + (xy + x + y)| = q\).

**Claim 2**

\[
|(B + (x + y)) \cap (B + (xy + x))| \geq q - k(k + 2m + 3)z. \tag{5.5}
\]

**Proof of Claim 2:** Let

\[
A' = B \cup (B + x) \cup (B + y) \cup (B + (x + y)) \cup (B + 2y) \cup (B + (x + 2y)), \\
C' = (B \cup (B + x)) - ((B + 2y) \cup (B + (x + 2y))).
\]

Similarly as in the proof of Claim 1, we have \(|C'| = |A'| - 4q \leq z\), and the following: (note that \(2(k + 1)y = xy + y\))

\[
|(B \cup (B + x)) - ((B + (xy + y)) \cup (B + (xy + x + y)))| \leq (k + 1)z.
\]

This implies that

\[
|(B \cup (B + x)) \cap ((B + (xy + y)) \cup (B + (xy + x + y)))| \geq 2q - (k + 1)z.
\]

Hence

\[
|(B \cup (B + x)) \cap (B + (xy + x + y))| \geq q - (k + 1)z. \tag{5.6}
\]

It follows from \((5.4)\) that

\[
|(B \cup (B + y)) \cap (B + (xy + x + y))| \geq q - (m + 1)z. \tag{5.7}
\]
As $B, (B + x), (B + y)$ are pairwise disjoint, we have

$$(B + (xy + x + y)) \cap (B + x) \subseteq (B + (xy + x + y)) - (B \cup (B + y)).$$

Hence, by (5.7),

$$|(B + (xy + x + y)) \cap (B + x)| \leq (m + 1)z.$$

Combining this with (5.6), we conclude that

$$|B \cap (B + (xy + x + y))| \geq q - (m + k + 2)z. \quad (5.8)$$

By adding $y$ to the involved sets of (5.4), we have

$$|((B+y) \cup (B+2y)) \cap ((B+(xy+x+y)) \cup (B+(xy+x+2y))| \geq 2q - (m + 1)z.$$

This implies that

$$|(B + (xy + x + y)) - ((B + y) \cup (B + 2y))| \leq (m + 1)z,$$

and hence

$$|((B + (xy + x + y)) \cap B) - ((B + y) \cup (B + 2y))| \leq (m + 1)z.$$

Note that $B \cap (B + y) = \emptyset$. Therefore

$$|((B + (xy + x + y)) \cap B) - (B + 2y)| \leq (m + 1)z. \quad (5.9)$$

Combining (5.8) and (5.9), one has

$$|(B + (xy + x + y)) \cap B \cap (B + 2y)| \geq q - (k + 2m + 3)z.$$

In particular,

$$|B \cap (B + 2y)| \geq q - (k + 2m + 3)z.$$

This implies that

$$|B - (B + 2y)| \leq (k + 2m + 3)z.$$
Note that $B + 2ky = B + (xy - y)$. So
\[
|B - (B + xy - y)| = |B - (B + 2ky)| \\
\leq \sum_{j=0}^{k-1} |(B + 2jy) - (B + 2(j+1)y)| \\
= \sum_{j=0}^{k-1} |(B - (B + 2y))| \\
\leq k(k + 2m + 3)z.
\]

This implies that
\[
|B \cap (B + (xy - y))| \geq q - k(k + 2m + 3)z. \tag{5.10}
\]

Hence, (5.5) is obtained by adding $x+y$ to the sets in (5.10). This completes the proof of Claim 2.

Because $B, B + y, B + (x + y)$ are pairwise disjoint, we conclude that
\[
q = |B + xy + x| \\
\geq |(B \cup (B + y)) \cap (B + (xy + x))| + |(B + (x + y)) \cap (B + (xy + x))| \\
\geq 2q - (k(k + 2m + 3) + (m + 1))z.
\]

Hence
\[
z \geq \frac{q}{k^2 + 2km + 3k + m + 1}.
\]

This completes the proof.

It is straightforward to verify that for $x \neq 1$, the lower bound for $\mu(M)$ given in Theorem 8 is bigger than $\beta(M)$ (referring to (1.2)). On the other hand, the upper bound for $\mu(M)$ given in Theorem 10 is bigger than $1/4$. Therefore, we conclude that $\mu(M) \neq 1/4, \beta(M)$ for sets $M$ of form (1.1).

6 Consequences and questions

Another parameter closely related to $\mu(M)$ is the circular chromatic number of the distance graph $G(Z, M)$. Suppose $H = (V, E)$ is a graph and $k \geq d$ are
positive integers. A \((k; d)\)-coloring of \(H\) is a mapping \(c : V \rightarrow \{0, 1, \cdots, k-1\}\) such that for every edge \(uv\) of \(H\), \(d \leq |c(u) - c(v)| \leq k - d\). The circular chromatic number \(\chi_c(H)\) of \(H\) is the minimum ratio \(k/d\) such that \(H\) has a \((k, d)\)-coloring. It is known [26] that for any graph \(G\), \([\chi_c(G)] = \chi(G)\) and
\[
\chi_f(G) \leq \chi_c(G) \leq \chi(G) = [\chi_c(G)]. \tag{6.1}
\]
It is also known [26] that
\[
\chi_c(G(Z, M)) \leq 1/\kappa(M). \tag{6.2}
\]

Based upon these connections, results in this article can be used to determine (or provide good bounds for) the values of \(\chi(G(Z, M)), \chi_c(G(Z, M))\) and \(\kappa(M)\) for almost difference closed sets \(M\). In this section, we give the exact values of \(\chi_c(G(Z, M))\) (and hence \(\chi(G(Z, M))\)) for all the almost difference closed sets \(M\), except for the sets \(M\) in (1.1). For the sets in (1.1), the bounds for \(\mu(M)\) proved in the previous section are sharp enough to determine \(\chi(G(Z, M))\). Moreover, we completely determine the value of \(\kappa(M)\) for any almost difference closed set \(M\). These results generalize some earlier work in those areas.

The results in this section indicate that the equalities \(\chi_f(G(Z, M)) = \chi_c(G(Z, M)) = 1/\kappa(M)\) hold for almost difference closed sets of Types A.1 and A.2, but not Type A.3.

Throughout this section, for any given set \(M\), we shall denote \(G(Z, M)\) by \(G\). We first consider the sets of Type A.1. The following is an immediate corollary of Theorem 5.

**Theorem 11** Suppose \(M = \{a, 2a, \cdots, (m - 1)a, b\}\) where \(gcd(a, b) = 1\).

- If \(a = 1\) and \(m \not\mid b\), then \(\chi_f(G) = \chi_c(G) = \chi(G) = m\).
- If \(a = 1\) and \(b = km\) for some integer \(k\), then
  \[\chi_f(G) = \chi_c(G) = m + 1/k, \quad \text{and} \quad \chi(G) = m + 1\.\]
- If \(a \geq 2\), then \(\chi_f(G) = \chi_c(G) = \chi(G) = m\).
For a special case $a = 1$ of Theorem 11 was proved in [3].

Secondly, we consider almost difference closed sets of Type A.2, $M = \{a, b, a + b\}$, where $a < b$, $b \neq 2a$, and $\gcd(a, b) = 1$. The values of $\kappa(M)$ for this type of sets $M$ were determined by Chen [6]. The chromatic number for the distance graphs $G = G(Z, M)$ was determined in [8, 21]. Before this article, the best known results on the circular chromatic number and fractional chromatic number of $G$ were obtained in [27]. To be precise, the following was proved in [6, 27]:

- If $b - a = 3k$, then $\chi_f(G) = \chi_c(G) = \kappa(M) = 3$.
- If $b - a = 3k + 1$, then
  \[
  3 + \frac{1}{a + 2k} \leq \chi_f(G) \leq \chi_c(G) \leq 3 + \frac{1}{a + k} = \beta(M) = \kappa(M).
  \]
- If $b - a = 3k + 2$, then
  \[
  3 + \frac{1}{b - k} \leq \chi_f(G) \leq \chi_c(G) \leq 3 + \frac{1}{b - k - 1} = \beta(M) = \kappa(M).
  \]

Using Theorem 6, we completely determine $\chi(G), \chi_f(G)$ and $\chi_c(G)$ for sets of Type A.2.

**Theorem 12** Suppose $M = \{a, b, a + b\}$ where $0 < a < b$, $\gcd(a, b) = 1$.

- If $b - a = 3k$, then $\chi_f(G) = \chi_c(G) = \chi(G) = 3$.

- If $b - a = 3k + 1$, then
  \[
  \chi_f(G) = \chi_c(G) = 3 + \frac{1}{a + k} \quad \text{and} \quad \chi(G) = 4.
  \]

- If $b - a = 3k + 2$, then
  \[
  \chi_f(G) = \chi_c(G) = 3 + \frac{1}{b - k - 1} \quad \text{and} \quad \chi(G) = 4.
  \]
The remaining of this section is devoted to almost difference closed sets of Type A.3. The chromatic number of the distance graph for the sets in type are determined in the following.

**Theorem 13** Suppose $M = \{x, y, y - x, x + y\}$, where $\gcd(x, y) = 1$. If $x, y$ are of distinct parity, then $\chi_f(G) = \chi_c(G) = \chi(G) = 4$, If $x$ and $y$ are both odd, then $\chi(G) = 5$.

For the special cases when $x, y$ are of distinct parity and when $x = 1$, the result of Theorem 13 was proved in [16]. For the case that both $x, y$ are odd, the conclusion $\chi(G) = 5$ follows from Theorem 10 and the fact that $\chi_f(G) \leq \chi(G) \leq 5$.

The relation between $\mu(M)$ and $\kappa(M)$ is of special interest. As remarked earlier, $\mu(M) \geq \kappa(M)$ for any $M$. The question whether or not the equality always hold was first raised in [2], and then discussed in [15]. An infinite family of sets $M$ for which $\mu(M) > \kappa(M)$ was given in [15].

Our results show that $\mu(M) = \kappa(M)$ for almost difference closed sets $M$ of Types A.1 and A.2. In contrast, if $M = \{x, y, y - x, x + y\}$ where $1 < x < y$, then the results below imply that $\mu(M) < \kappa(M)$.

It is known and not hard to show (cf. [15]) that $\kappa(M)$ is a fraction whose denominator always divides the sum of some pair of elements in $M$. Indeed, suppose $\kappa(M) = |tM| = p/q$, then there exist $a, b \in M$ such that $at = k_1 + p/q$ and $bt = k_2 - p/q$, for some integers $k_1$ and $k_2$ (otherwise, one may increase or decrease $t$ by a small amount so that $|tM|$ increases). This implies that $t = (k_1 + k_2)/(a + b)$, and hence $q|(a + b)$.

**Theorem 14** If $M = \{x, y, y - x, y + x\}$, where one of $x$ and $y$ is a multiple of 4 and the other is odd, then $\kappa(M) < 1/4 = \mu(M)$.

**Proof.** By Theorem 7, it remains to show that $\kappa(M) \neq 1/4$.

If $x$ is a multiple of 4 and $y$ is odd, then the sum of any two numbers in $M$ is not a multiple of 4. Hence $\kappa(M) \neq 1/4$.

If $y$ is a multiple of 4 and $x$ is odd, suppose to the contrary that $\kappa(M) = 1/4$. Because $x$ and $y - x$ is the only possible pair of elements in $M$ such that
4 divides their sum, so there exists some real number \( t \), \( t = (k_1 + k_2)/(x + (y - x)) \), such that \( ||tM|| = 1/4 \). This is impossible, since then \( ||ty|| = 0 \).

Indeed, the exact value of \( \kappa(M) \) for the case \( M = \{x, y, y - x, y + x\} \), where one of \( x \) and \( y \) is a multiple of 4, can be completely determined as follows.

**Theorem 15** Suppose \( M = \{x, y, y - x, y + x\} \), where \( \gcd(x, y) = 1 \), and one of \( x \) and \( y \) is a multiple of 4. Then \( \kappa(M) = \beta(M) = \phi_4(n)/n \), where

\[
    n = \begin{cases} 
        2y + x, & \text{if } x \equiv 0 \pmod{4} \text{ and } y \equiv 3 \pmod{4}, \text{ or } \\
        x \equiv 1 \pmod{4} \text{ and } y \equiv 0 \pmod{4}, \\
        2x + y, & \text{if } x \equiv 0 \pmod{4} \text{ and } y \equiv 1 \pmod{4}, \\
        2y - x, & \text{if } x \equiv 3 \pmod{4} \text{ and } y \equiv 0 \pmod{4}. 
    \end{cases}
\]

**Proof.** By the discussion preceding Theorem 14, \( \kappa(M) = p/q \) for some \( p, q \) such that \( q \) divides the sum of two elements of \( M \). By Theorem 14, \( p/q < 1/4 \). Hence \( \kappa(M) \leq \beta(M) \).

It is straightforward to verify that the \( n \) values chosen satisfy \( \beta(M) = \phi_4(n)/n \). Therefore, it suffices to show that \( \kappa(M) \geq \phi_4(n)/n \) for the chosen \( n \).

**Case 1:** \( x \equiv 0 \pmod{4} \) and \( y \equiv 3 \pmod{4} \)

Let \( a = \phi_4(2y + x) \). Note that \( \gcd(y, 2y + x) = \gcd(x, y) = 1 \). Hence there exist \( k \) and \( k' \) such that \( yk = (2y + x)k' - a \). This implies

\[
    (y + x)k = (2y + x)k' - yk = (2y + x)(k - k') + a,
\]

\[
    xk = (2y + x)(k - 2k') + 2a, \text{ and }
\]

\[
    (y - x)k = (2y + x)(k' - k) - 3a.
\]

Set \( t = k/(2y + x) \). Then we get \( ||ty|| = ||t(x + y)|| = a/(2y + x) \), and \( ||yk||, ||(y - x)k|| \geq a/(2x + y) \). Hence \( \kappa(M) \geq \frac{\phi_4(2y + x)}{2y + x} \).

**Case 2:** \( y \equiv 0 \pmod{4} \) and \( x \equiv 1 \pmod{4} \)

This case is symmetric to Case 1. We shall omit the details.

**Case 3:** \( x \equiv 0 \pmod{4} \) and \( y \equiv 1 \pmod{4} \)

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Let \( a = \phi_4(2x + y) \). Since \( \gcd(x, 2x + y) = \gcd(y, 2x + y) = \gcd(x, y) = 1 \), there exist integers \( k \) and \( k' \) such that \( xk = (2x + y)k' - a \), where \( a = \phi_4(2x + y) \). This implies
\[
(x + y)k = (2x + y)k - xk = (2x + y)(k - k') + a,
\]
\[
yk = (2x + y)(k - 2k') + 2a, \quad \text{and}
\]
\[
(y - x)k = (2x + y)(k - 3k') + 3a.
\]

Set \( t = k/(2x + y) \). Then we get \( ||tM|| = a/(2x + y) \). Hence \( \kappa(M) \geq \frac{\phi_4(2x + y)}{2x + y} \).

**Case 4: \( x \equiv 3 \pmod{4} \) and \( y \equiv 0 \pmod{4} \)**

Let \( a = \phi_4(2y - x) \). Since \( \gcd(y - x, 2y - x) = 1 \), there exist some integers \( k \) and \( k' \) such that \( (y - x)k = (2y - x)k' - a \). This implies
\[
yk = (2y - x)(k - k') + a,
\]
\[
xk = (2y - x)(k - 2k') + 2a, \quad \text{and}
\]
\[
(y - x)k = (2y + x)(2k - 3k') + 3a.
\]

Let \( t = k/(2y - x) \). Then \( ||tM|| = a/(2y - x) \). Hence \( \kappa(M) \geq \frac{\phi_4(2y - x)}{2y - x} \).

If \( M = \{x, y, y - x, x + y\} \), \( x, y \) are of distinct parity and none of \( x, y \) is a multiple of 4, then \( ||\frac{1}{4}M|| = 1/4 \), so \( \kappa(M) = 1/4 \). Hence by Theorem 7, we have \( \mu(M) = \kappa(M) \).

It remains to consider the case that \( M = \{x, y, y - x, x + y\} \), where \( x, y \) are both odd (i.e. sets in (1.1)). The result below determines the values of \( \kappa(M) \) for such sets. Note that by Theorem 10, \( \mu(M) < 1/4 \). Hence \( \kappa(M) \leq \mu(M) < 1/4 \).

**Theorem 16** If \( M = \{x, y, y - x, x + y\} \), where \( x \) and \( y \) are both odd, then \( \kappa(M) = \beta(M) = \phi_4(n)/n \), where

\[
n = \begin{cases} 
2y - x, & \text{if } x \equiv y \equiv 1 \pmod{4}, \text{ or} \\
x \equiv 1 \pmod{4}, y \equiv 3 \pmod{4}, \text{ and } y \geq 3x, \\
2y + x, & \text{if } x \equiv 3 \pmod{4}, \\
2x + y, & \text{if } x \equiv 1 \pmod{4}, y \equiv 3 \pmod{4}, \text{ and } y < 3x.
\end{cases}
\]
The proof of Theorem 16 is similar to that of Theorem 15, and is omitted. The following is an immediate corollary of Theorems 8 and 16.

**Theorem 17** If $M = \{x, y, y - x, y + x\}$, where $x$ and $y$ are both odd, and $x \neq 1$, then $\kappa(M) < \mu(M)$.

For the case that $x = 1$ (i.e. $M = \{1, y, y - 1, y + 1\}$ for some odd $y$), Theorems 16 and 9 imply that $\mu(M) = \kappa(M)$.

For the sets that are not almost difference closed, the values of $\mu(M)$ or $\kappa(M)$ are known only for very few sets of $M$. In particular, the following question remains open:

**Question 1** Suppose $M$ is a set consisting of three positive integers. Is it true that $\mu(M) = \kappa(M)$?

The parameter $\kappa(M)$ has also interesting connections to Diophantine approximations [5, 7, 23], view obstruction [9], as well as to problems concerning flows and colorings of graphs. [1, 27]. We refer the readers to the related references for these interesting connections. A long standing open question concerning $\kappa(M)$ is the following conjecture due to Wills [23].

**Conjecture 3** Suppose $M$ is a finite set of positive integers with $|M| = m$. Then $\kappa(M) \geq 1/(m + 1)$.

Conjecture 3 is also called the “lonely runner conjecture” by Goddyn et al. [1] due to the following interpretation: Suppose $m$ runners run laps on a circular track of unit length. Each runner maintains a distinct constant speed. A runner is called lonely if the distance (on the circular track) between him (or her) and every other runner is at least $1/k$. The conjecture is equivalent to assert that for each runner, there is a time when (s)he is lonely.

Although Conjecture 3 has attracted considerable attention, it remains open for $m \geq 5$. For $m \leq 4$, the conjecture is confirmed [1, 5, 7, 9].

If Conjecture 3 is true, then the bound $1/(m + 1)$ is sharp. In addition to difference closed sets $M$, only very few known sets attain this bound. (This
is another motivation we consider almost difference closed sets). As noted earlier, it is easy to see that \( \mu(M) = 1/\chi_f(G) \geq 1/(m + 1) \). This bound is attained by difference closed sets \( M \). However, we do not know whether there is any other set \( M \) for which this bound is also attained. Intuitively, \( \chi_f(G) \) is more likely to be small (and hence \( \mu(M) \) is big) if the clique size of \( G \) is small. So, it seems natural to conjecture the following.

**Conjecture 4** Suppose \( M \) is a finite set of positive integers with \( |M| = m \). If \( M \) is not almost difference closed, then \( \chi_f(G) \leq m \), or equivalently, \( \mu(M) \geq 1/m \).

The conjecture above is weaker than the following conjecture of [27]:

**Conjecture 5** Suppose \( M \) is a finite set of positive integers with \( |M| = m \). If \( M \) is not almost difference closed, then \( \chi(G(Z, M)) \leq m \).

We note here that for \( m \leq 3 \), both Conjectures 4 and 5 are true [5, 7, 27].

**References**


