Multi-level Distance labelings for Trees

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Abstract

For a graph $G$, we denote its diameter by $\text{diam}(G)$, and denote the distance between any two vertices, $u$ and $v$, by $d_G(u, v)$. A multi-level distance labeling (or radio labeling) of $G$ is a function $f$ that assigns to each vertex a non-negative integer such that for any pair of vertices $u$ and $v$, it is satisfied that $|f(u) - f(v)| \geq \text{diam}(G) - d_G(u, v) + 1$. The span of $f$ is $\{\max f(V) - \min f(V)\}$. The radio number of $G$ is the minimum span of a radio labeling for $G$. We prove a lower bound for the radio number of trees, and characterize the trees achieving this bound. Moreover, we prove another lower bound for the radio number of spiders (trees with at most one vertex of degree more than two) in terms of the lengths of their legs, and characterize the spiders achieving this bound. Our results generalize the radio number for paths obtained by Liu and Zhu [17].

1 Introduction

Multiple-level distance labeling can be regarded as an extension of distance two labeling, and both of them are motivated by the channel assignment problem introduced by Hale [10]. Given a set of stations (or transmitters),

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a valid channel assignment is a function that assigns to each transmitter or station with a channel (non-negative integer), such that the interference is avoided. The task is to find a valid channel assignment with the minimum span of the channels used. The degree (or level) of interference is related to the locations of the stations – the closer of two stations, the stronger interference that might occur. In order to avoid interference, the separation between the channels assigned to a pair of close stations must be “large” (depending on the distance between the two stations) enough.

A graph model for this problem is to represent each station by a vertex, and connect two close stations by an edge. Let $G$ be a connected graph. We denote the distance (length of a shortest path) between two vertices, $u$ and $v$, by $d_G(u, v)$, or $d(u, v)$ if $G$ is clear in the context. For a graph $G$, a distance two labeling (motivated by the channel assignment problem with two levels of interference) is a function $f : V \rightarrow \{0, 1, 2, 3, \cdots\}$ such that the following are satisfied:

\[ |f(u) - f(v)| \geq \begin{cases} 
2, & \text{if } d(u, v) = 1; \\
1, & \text{if } d(u, v) = 2. 
\end{cases} \]

The span of such a function $f$ is defined by \{max $f(V) - \min f(V)$\}. The $\lambda$-number for a graph $G$, denoted by $\lambda(G)$, is the minimum span of a distance two labeling for $G$. Distance two labeling has been studied intensively in the past fifteen years (cf. [1, 2, 5 - 9, 11 - 16, 18, 19]).

The diameter of a graph $G$, denoted by diam($G$), is the maximum distance over all pairs of vertices. Multi-level distance labeling is motivated by the channel assignment problem with diam($G$) levels of interference for the corresponding graph $G$. Hence, a multi-level distance labeling (or distance labeling for short) is a function $f : V(G) \rightarrow \{0, 1, 2, 3, \cdots\}$, so that the following is satisfied for any vertices $u$ and $v$:

\[ |f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1. \]

The radio number (as suggested by the AM/FM radio channel assignment [4]) for a graph $G$, denoted by $rn(G)$, is the minimum span of a distance
labeling for $G$. A distance labeling with span equal to the radio number of a graph $G$ is called an \emph{optimal distance labeling}. Note that when $\text{diam}(G) = 2$, distance two labeling coincides with multi-level distance labeling, and in this case, $\lambda(G) = \text{rn}(G)$.

Determining the radio number seems a difficult task, even for some basic graphs. For instance, the radio number for paths and cycles has been studied by Chartrand et al. [4, 3], in which, bounds of the radio numbers for paths and cycles were presented. Later on, the radio number for paths and cycles were completely settled by Liu and Zhu [17].

We investigate the radio number for trees. Using the terminology of rooted trees and level functions, we prove a lower bound for the radio number of trees, and characterize the trees achieving this bound. Expanding this method further, we study a special family of trees called spiders. A \emph{spider} is a tree with at most one vertex of degree more than two. Besides applying the results on trees to spiders, we present in Section 3 another lower bound for the radio number of spiders, and in Section 4, we characterize the spiders achieving this bound.

## 2 Radio Number for Trees

Let $T$ be a tree rooted at a vertex $w$. For any two vertices $u$ and $v$, if $u$ is on the $(w, v)$-path, we call $u$ an \emph{ancestor} of $v$, and $v$ a \emph{descendant} of $u$. Note, the root $w$ is an ancestor of every vertex; and every vertex is its own ancestor and descendant. Fix any $w$ as the root, define the \emph{level function} on $V(T)$ by

$$L_w(u) = d(w, u), \text{ for any } u \in V(T).$$

And, for any $u, v \in V(T)$, define

$$\phi_w(u, v) = \max \{L_w(t) : t \text{ is a common ancestor of } u \text{ and } v\}.$$

Let $w'$ be a neighbor of $w$. We call the subtree induced by $w$ together with all the descendents of $w'$ a \emph{branch}. 

Observation 1  Let $T$ be a tree rooted at $w$. For any two vertices $u$ and $v$,

(1) $\phi_w(u, v) = 0$ if and only if $u$ and $v$ belong to different branches (unless one of them is $w$), and

(2) $d(u, v) = L_w(u) + L_w(v) - 2\phi_w(u, v)$.

For any vertex $w$ in a tree $T$, the weight of $T$ (rooted) at $w$ is defined by:

$$w_T(w) = \sum_{u \in V(T)} L_w(u).$$

The weight of $T$ is the smallest weight among all vertices of $T$:

$$w(T) = \min \{w_T(w) : w \in V(T)\}.$$  

A vertex $w^*$ of a tree $T$ is called a weight center of $T$ if $w_T(w^*) = w(T)$.

Lemma 1  Suppose $w^*$ is a weight center of an $m$-vertex tree $T$. Then each component of $T - w^*$ contains at most $m/2$ vertices.

Proof. Suppose to the contrary that there exists a component $F$ of $T - w^*$ with $|V(F)| > m/2$. Let $w \in V(F)$ be a neighbor of $w^*$. Then $w_T(w) = w_T(w^*) - |V(F)| + |V(T - F)| < w_T(w^*)$, since when we shift the root from $w^*$ to $w$, the level decreases by one for each vertex in $F$, and increases by one for every other vertex. This contradicts the assumption that $w^*$ is a weight center. \hfill \square

Theorem 2  Every tree $T$ has at most two weight centers. If $|V(T)|$ is odd, then $T$ has a unique weight center. If $|V(T)|$ is even, then $T$ has two weight centers, say $w$ and $w'$, if and only if $ww' \in E(T)$ and the deletion of $ww'$ from $T$ results in two equal-sized components.

Proof. Assume $|V(T)| \geq 3$ (it is trivial, otherwise). Suppose $w^*$ is a weight center of $T$. Let $F_1, F_2, \ldots, F_k$ be the components of $T - w^*$, and let $v_1, v_2, \ldots, v_k$
be the neighbors of \( w^* \) with \( v_i \in F_i \). By Lemma 1, \( k \geq 2 \), and \(|V(F_i)| \leq |V(T)|/2 \) for all \( i \). Let \( v \in V(T) - \{w^*, v_1, v_2, \ldots, v_k\} \). We claim that \( v \) can not be a weight center. Assume \( v \in F_j \) for some \( j \). Then in \( T-v \), there is a component containing \( v_j \) and \( V(T-F_j) \), with more than \(|V(T)|/2 \) vertices (since \(|V(T-F_j)| \geq |V(T)|/2 \)). By Lemma 1, \( v \) is not a weight center.

Assume \( v_j \) is a weight center besides \( w^* \), for some \( j \). Then \( T-v_j \) is a component in \( T-v_j \). By Lemma 1, \(|V(T-F_j)| \leq |V(T)|/2 \). Since \(|V(F_j)| \leq |V(T)|/2 \), it must be that \(|V(T-F_j)| = |V(F_j)| = |V(T)|/2 \). So \(|V(T)| \) is even. It is easy to see that if \( v_j \) is a weight center, then \( v_i \) can not be a weight center, for \( i \neq j \). This completes one direction of the proof. It remains to show that if there exists an edge \( ww' \) in \( T \) such that the deletion of \( ww' \) from \( T \) results in two equal-sized components, then \( w \) and \( w' \) are both weight centers of \( T \). Assume such an edge \( ww' \) exists. It is clear that \( w_T(w) = w_T(w') \). Moreover, by the same discussion in the above, \( w \) and \( w' \) are the only two vertices that would satisfy the conclusion of Lemma 1, hence they are the only weight centers of \( T \).

As we are seeking for the minimum span of a distance labeling for a graph \( G \), without loss of generality, throughout the article we assume that the label 0 is used by any distance labeling. A distance labeling is a one-to-one function. On the other hand, any one-to-one function \( f \) on \( V(G) \), with \( 0 \in f(V) \), induces an ordering of \( V(G) \), which is a line-up of the vertices with increasing labels. We denote this ordering by \( U(f) \), where

\[
V(G) = U(f) = (u_0, u_1, u_2, \ldots, u_{|V|-1}), \text{ with }
0 = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{|V|-1}).
\]

If \( f \) is a distance labeling, then the span of \( f \) is \( f(u_{|V|-1}) \).

**Proposition 3** If \( f \) is a distance labeling of span \( k \) for a graph \( G \), then the function \( f^* \) on \( V(G) \) defined by \( f^*(v) = k - f(v) \) is also a distance labeling of span \( k \) for \( G \).
Proposition 3 implies that for a distance labeling $f$ and its corresponding ordering $U(f)$, the reverse of the ordering, $(u_{|V|-1}, \ldots, u_2, u_1, u_0)$, is the ordering of another distance labeling $f^*$ with the same span as $f$. We call the vertices $u_0, u_{|V|-1}$ the ends of $f$.

**Theorem 4** Let $T$ be an $m$-vertex tree with diameter $d$. Then

$$rn(T) \geq (m - 1)(d + 1) + 1 - 2w(T).$$

Moreover, the equality holds if and only if for every weight center $w^*$, there exists a distance labeling $f$ with $f(u_0) = 0 < f(u_1) < \cdots < f(u_{m-1})$, where all the following hold (for all $0 \leq i \leq m - 2$):

1. $u_i$ and $u_{i+1}$ belong to different branches (unless one of them is $w$);
2. $\{u_0, u_{m-1}\} = \{w, v\}$, where $v$ is some vertex with $L_{w^*}(v) = 1$;
3. $f(u_{i+1}) = f(u_i) + d + 1 - L_{w^*}(u_i) - L_{w^*}(u_{i+1})$.

**Proof.** Let $f$ be an optimal distance labeling for $T$, where $f(u_0) = 0 < f(u_1) < f(u_2) < \cdots < f(u_{m-1})$. Then $f(u_{i+1}) - f(u_i) \geq (d + 1) - d(u_{i+1}, u_i)$ for all $0 \leq i \leq m - 2$. Summing up these $m - 1$ inequalities, we get

$$rn(T) = f(u_{m-1}) \geq (m - 1)(d + 1) - \sum_{i=0}^{m-2} d(u_{i+1}, u_i). \quad (2.1)$$

Let $w^*$ be a weight center. Each vertex of $T$ occurs exactly twice in the summation term in the above, except $u_0$ and $u_{m-1}$, for which each occurs exactly once. Hence, by Observation 1, we get

$$\sum_{i=0}^{m-2} d(u_{i+1}, u_i) = 2 \left( \sum_{u \in V(T)} L_{w^*}(u) \right) - L_{w^*}(u_0) - L_{w^*}(u_{m-1}) - 2 \sum_{i=0}^{m-2} \phi_{w^*}(u_{i+1}, u_i)$$

$$\leq 2 \left( \sum_{u \in V(T)} L_{w^*}(u) \right) - 1 = 2w(T) - 1. \quad (2.2)$$

By (2.1) and (2.2), the lower bound for $rn(T)$ is obtained. Note, the inequality in (2.2) becomes equality if and only if $\phi_{w^*}(u_{i+1}, u_i) = 0$ for all $i$, and $\{u_0, u_{m-2}\} = \{w, v\}$ for some $v$ with $L_{w^*}(v) = 1$.  

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To prove the moreover part, assume that $rn(T)$ achieves the lower bound. Let $w^*$ be a weight center, and let $f$ be an optimal distance labeling with $f(u_0) = 0 < f(u_1) < f(u_2) < \cdots < f(u_{m-1}) = rn(T)$. Since $rn(T) = (m-1)(d+1) + 1 - 2 \sum_{u \in V(T)} L_{w^*}(u)$, by the above discussion, both inequalities in (2.1) and (2.2) become equalities, implying (1) and (2). In addition, by the equality of (2.1), we have $f(u_{i+1}) - f(u_i) = d + 1 - d(u_{i+1}, u_i)$ for all $i$. So, (3) holds.

To prove the converse, let $w^*$ be a weight center. Suppose there exists a distance labeling $f$ such that (1 - 3) hold. By (2) and (3), $f(v_{m-1}) = (m-1)(d+1) + 1 - 2 \sum_{u \in V(T)} L_{w^*}(u) = (m-1)(d+1) + 1 - 2w(T)$. Hence $rn(T) \leq f(v_{m-1}) = (m-1)(d+1) + 1 - 2w(T) \leq rn(T)$. Therefore, $rn(T)$ achieves the lower bound.

Immediate consequences of Theorem 4 include the radio number for paths (which was settled in [17] by a different approach). The weight center for an odd path $P_{2k+1}$, for some $k \geq 2$, is the middle vertex. The radio number for $P_{2k+1}$ is larger than the bound shown in Theorem 4, since there does not exist a distance labeling $f$ that satisfies (1 - 3) in Theorem 4. It is not hard to find a distance labeling for $P_{2k+1}$ with span one more than the bound of Theorem 4 (cf. [17] or see Section 4), hence $rn(P_{2k+1})$ is obtained.

Even paths $P_{2k}$ have radio numbers equal to the bound in Theorem 4, as one can find a distance labeling satisfying (1 - 3) in Theorem 4 (cf. [17] or see Section 4). Other than even paths, there are many trees whose radio numbers achieve this bound. See Figure 1 for some examples.

3 Lower Bounds for Spiders

For a spider, if there exists a vertex of degree more than two, then we call that vertex the *center*. If every vertex has degree at most two, then it is a path. The center for an odd path is the middle vertex, and for an even path
Figure 1: Optimal distance labelings for trees with radio numbers achieving the bound of Theorem 4.

is one of the two middle vertices. A leg of a spider is a path with one end at the center and the other at one of the leaves (degree-one vertices). We denote a spider with \( n \) legs \( (n \geq 2) \) by \( S_{l_1,l_2,l_3,...,l_n} \), where each \( l_i \in \mathbb{Z}^+ \) is the length (number of edges) of each leg, with \( l_1 \geq l_2 \geq \cdots \geq l_n \). The center is not always a weight center. By Theorem 2, the center of a spider is a weight center if and only if \( l_1 \leq |V(G)|/2 \).

Let \( G \) be a spider, \( G = S_{l_1,l_2,...,l_n} \). The center of \( G \) is denoted by \( v_{0,0} \); the vertex set of \( G \) is denoted by

\[
V(G) = V_1 \cup V_2 \cdots \cup V_n,
\]

where each \( V_i \) is the vertex set of the \( i \)-th leg with length \( l_i \), that is, assuming \( v_{i,0} = v_{0,0} \),

\[
V_i = \{v_{i,j} : 0 \leq j \leq l_i \}, \text{ where } \{v_{i,j}, v_{i,j+1}\} \in E(G), \ 0 \leq j \leq l_i - 1.
\]

The level function for \( G \) (rooted) at \( v_{0,0} \) is denoted by \( L \). That is, \( L(v_{i,j}) = j \), for any vertex \( v_{i,j} \). The number of vertices in \( G \) is \( l_1 + l_2 + \cdots + l_n + 1 \), and the diameter of \( G \) is \( l_1 + l_2 \).
Theorem 5 Let $G = S_{l_1,l_2,\ldots,l_n}$ be a spider. Then

$$w(G) = (1/2) \sum_{i=1}^{n} l_i(l_i+1) - \left[ \frac{l_1 - (l_2 + \cdots + l_n - 1)}{2} \right] \left[ \frac{l_1 - (l_2 + \cdots + l_n + 1)}{2} \right],$$

where the last term becomes 0 if $l_1 \leq l_2 + l_3 + \cdots + l_n + 1$.

Proof. Let $m = |V(G)| = l_1 + l_2 + \cdots + l_n + 1$. By definition, $w_G(v_{0,0}) = (1/2) \sum_{i=1}^{n} l_i(l_i+1)$. By Lemma 1, $v_{0,0}$ is a weight center of $G$ if and only if $l_1 \leq m/2$. This proves the case that $l_1 \leq l_2 + l_3 + \cdots + l_n + 1$.

Assume $l_1 > l_2 + l_3 + \cdots + l_n + 1$. By Lemma 1 and Theorem 2, we may assume that a weight center of $G$ is $v_{1,k}$, where $k = l_1 - [m/2]$. Then

$$w(G) = w_G(v_{1,k}) = w_G(v_{0,0}) - k(l_1 - k) + k(m - l_1 - 1),$$

since if we switch the root from $v_{0,0}$ to $v_{1,k}$, the level decreases by $k$ for each $v_{1,k'}$, $k + 1 \leq k' \leq l_1$, increases by $k$ for each vertex in $\{V(G) - V_1\}$; and the sums of levels for the vertices $v_{0,0}$ and $v_{1,i}$, $1 \leq i \leq k$, are the same. The result then follows by some calculation.

By Theorems 4 and 5, we obtain

Corollary 6 Let $G = S_{l_1,l_2,\ldots,l_n}$ be a spider. Then

$$rn(G) \geq \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + 1 + 2\left[ \frac{l_1 - (l_2 + \cdots + l_n - 1)}{2} \right] \left[ \frac{l_1 - (l_2 + \cdots + l_n + 1)}{2} \right],$$

where the last term becomes 0 if $l_1 \leq l_2 + \cdots + l_n + 1$.

In the rest of this section, we prove another lower bound for the radio number of spiders, which in some cases, is better than the one in Corollary 6. Note that an easy case is when $l_1 = 1$, which is a star $S_{1,1,\ldots,1}$. It is easy to see that $rn(S_{1,1,\ldots,1}) = n + 1$. Hence, we assume $l_1 \geq 2$.

Observation 2 In a spider $S_{l_1,l_2,\ldots,l_n}$, the distance between any two vertices has

$$d(v_{i,j}, v_{i',j'}) = \begin{cases} j + j' & \text{if } i \neq i'; \\ |j - j'| = \max\{j, j'\} - \min\{j, j'\} & \text{if } i = i'. \end{cases}$$
For any distance labeling \( f \) of \( G \), we adopt the same notation \( U(f) = \{u_0, u_1, \ldots, u_{|V(G)|-1}\} \) used in the previous section. Moreover, for any \( 0 \leq i \leq |V| - 2 \), set

\[
x_i = f(u_{i+1}) - f(u_i) + L(u_{i+1}) + L(u_i) - \text{diam}(G) - 1.
\]

For integers \( 0 \leq i < j \leq |V| - 1 \), \( \{u_i, u_{i+1}, u_{i+2}, \ldots, u_j\} \) (respectively, \( \{f(u_i), f(u_{i+1}), \ldots, f(u_j)\} \)) is called a set of consecutive vertices (respectively, consecutive labels).

Let \( f \) be a distance labeling for a spider \( G = S_{l_1, l_2, \ldots, l_n} \). By Observation 2 and definition of distance labeling, \( x_i \geq 0 \) for any \( 0 \leq i \leq |V| - 2 \). Moreover, if \( u_{i+1}, u_i \in V_k \) for some \( k \), then \( x_i \geq 2 \min \{L(u_{i+1}), L(u_i)\} \).

**Lemma 7** Let \( G = S_{l_1, l_2, \ldots, l_n} \). Suppose \( f \) is a non-negative integral one-to-one function on \( V(G) \) with the ordering of \( V(G) = U(f) = (u_0, u_1, u_2, \ldots, u_{|V|-1}) \). Then \( f \) is a distance labeling for \( G \) if and only if the following hold for any set of consecutive vertices \( \{u_i, u_{i+1}, u_{i+2}, \ldots, u_j\} \), \( 0 \leq i < j \leq |V|-1 \):

\[
\text{(1)} \quad \sum_{t=i}^{j-1} x_t \geq 2 \left( \sum_{t=i+1}^{j-1} L(u_t) \right) - (j - i - 1)(l_1 + l_2 + 1).
\]

\[
\text{(2)} \quad \text{If } u_i, u_j \in V_k \text{ for some } k, \text{ then}
\]

\[
\sum_{t=i}^{j-1} x_t \geq 2 \left( \sum_{t=i+1}^{j-1} L(u_t) \right) - (j - i - 1)(l_1 + l_2 + 1) + 2 \min \{L(u_i), L(u_j)\}.
\]

**Proof.** Suppose \( f \) is a distance labeling for \( G \). Since \( \text{diam}(G) = l_1 + l_2 \), summing up the definition of \( x_t \) for \( i \leq t \leq j - 1 \), we get

\[
\sum_{t=i}^{j-1} x_t = f(u_j) - f(u_i) - (j - i)(l_1 + l_2 + 1) + 2 \left( \sum_{t=i+1}^{j-1} L(u_t) \right) + L(u_i) + L(u_j).
\]

As \( f \) is a distance labeling, by Observation 2, we have

\[
f(u_j) - f(u_i) \geq l_1 + l_2 + 1 - L(u_j) - L(u_i).
\]
Therefore, (1) follows.

To prove (2), again by definition and Observation 2, we have

\[
f(u_j) - f(u_i) = (j - i)(l_1 + l_2 + 1) - 2 \left( \sum_{t=i+1}^{j-1} L(u_t) \right) - L(u_i) - L(u_j) + \sum_{t=i}^{j-1} x_t
\]
\[
\geq l_1 + l_2 + 1 - \max\{L(u_j), L(u_i)\} + \min\{L(u_j), L(u_i)\}.
\]

Hence, (2) follows by easy calculation.

To prove the converse, assume \(f\) satisfies (1) and (2). To show that \(f\) is a distance labeling, it suffices to verify the following inequality for any \(0 \leq i < j \leq |V| - 1\),

\[
f(u_j) - f(u_i) \geq l_1 + l_2 + 1 - d(u_i, u_j). \quad (3.1)
\]

If \(u_i\) and \(u_j\) belong to different legs, then \(d(u_i, u_j) = L(u_i) + L(u_j)\). By (1) and Observation 2, equation (3.1) holds. If \(u_i, u_j \in V_k\) for some \(k\), then (3.1) follows by (2) and Observation 2.

We introduce a few more notations. Throughout the article, for a spider \(G = S_{l_1, l_2, \ldots, l_n}\) with \(l_1 - l_2 \geq 2\), let

\[
z = \left\lfloor \frac{l_1 - l_2 - 2}{2} \right\rfloor.
\]

Suppose \(f\) is a distance labeling for a spider with \(l_1 - l_2 \geq 2\). For \(j = 0, 1, 2, \ldots, z\), let \(t_j\) be the integers, \(0 \leq t_j \leq |V| - 1\), with

\[
u_{t_j} = v_{1, l_1 - j}.
\]

**Lemma 8** Let \(f\) be a distance labeling for \(G = S_{l_1, l_2, \ldots, l_n}\), where \(l_1 - l_2 \geq 2\). If \(1 \leq t_j \leq |V| - 2\) for some \(j = 0, 1, \ldots, z\), then

\[
x_{t_j-1} + x_{t_j} \geq l_1 - l_2 - (2j + 1) \geq 1.
\]

Moreover, the first equality holds only if \(u_{t_j-1}\) and \(u_{t_j+1}\) belong to different legs, unless one of them is the center.
Proof. Let $j = 0, 1, 2, \cdots, z$. Assume $v_{1, i_j - j} = u_{t_j}$ for some $1 \leq t_j \leq |V| - 2$. Consider the three consecutive vertices $\{u_{t_j - 1}, u_{t_j}, u_{t_j + 1}\}$. Since $L(u_{t_j}) = l_1 - j$, by Lemma 7 (1), we have

$$x_{t_j} + x_{t_j - 1} \geq 2L(u_{t_j}) - (l_1 + l_2 + 1) = l_1 - l_2 - 2j - 1.$$ 

The result then follows, as $0 \leq j \leq \lfloor \frac{|V| - 2}{2} \rfloor$ and $l_1 - l_2 \geq 2$.

To prove the moreover part, assume $u_{t_j - 1}, u_{t_j + 1} \in V_k - \{v_{0,0}\}$ for some $k$. Then by Lemma 7 (2), $x_{t_j} + x_{t_j - 1} \geq 2L(u_{t_j}) - (l_1 + l_2 + 1) + 2 > l_1 - l_2 - 2j - 1$.

Lemma 9. Let $f$ be a distance labeling for $G = S_{l_1, l_2, \ldots, l_0}$. If there exist some $0 \leq j, j' \leq z$, such that $t_{j'} = t_j + 1$ (that is, $v_{1, i_j - j}$ and $v_{1, i_j - j'}$ are consecutive), then

$$x_{t_j} > 2(l_1 - l_2 - j' - j - 1) = 2l_1 - l_2 - (2j' + 1) - (2j + 1).$$

Proof. By Lemma 7 (2),

$$x_{t_j} \geq 2 \min \{L(u_{t_j}), L(u_{t_j'})\} = 2 \min \{l_1 - j, l_1 - j'\} > 2(l_1 - l_2 - j' - j - 1).$$

Lemma 10. Let $f$ be a distance labeling for $G = S_{l_1, l_2, \ldots, l_0}$. If $l_1 - l_2 \leq 1$, or $l_1 - l_2 \geq 2$ and $1 \leq t_j \leq |V| - 2$ for all $j = 0, 1, \cdots, z$, then

$$\sum_{i=0}^{\lfloor \frac{|V| - 2}{2} \rfloor} x_i \geq \lfloor \frac{l_1 - l_2}{2} \rfloor \lfloor \frac{l_1 - l_2}{2} \rfloor.$$ 

Moreover, if $l_1 - l_2 \geq 2$, then the equality holds only if $u_{t_j - 1}$ and $u_{t_j + 1}$ belong to different legs (unless one of them is the center) for all $1 \leq j \leq z$, and $v_{1, i_j - j}$ and $v_{1, i_{j'} - j'}$ are not consecutive for any $0 \leq j < j' \leq z$. 

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Proof. It is trivial if \( l_1 - l_2 \leq 1 \), as \( x_i \geq 0 \) for all \( i \).

Assume \( l_1 - l_2 \geq 2 \). By Lemmas 8 and 9, we have

\[
\sum_{i=0}^{\lfloor V \rfloor - 2} x_i \geq (l_1 - l_2)(z + 1) - \sum_{j=0}^{z}(2j + 1)
= (l_1 - l_2)(z + 1) - (z + 1)^2
= \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor.
\]

The moreover part follows by Lemma 9, and the moreover part of Lemma 8.

Lemma 11 Let \( f \) be a distance labeling for \( G = S_{l_1,l_2,...,l_n} \) with ordering of \( V(G) = U(f) = (u_0, u_1, u_2, \ldots, u_{\lfloor V \rfloor - 1}) \). Then

\[
2 \left( \sum_{i=1}^{\lfloor W \rfloor - 2} L(u_i) \right) + L(u_0) + L(u_{\lfloor W \rfloor - 1}) \leq \sum_{i=1}^{n} l_i(l_i + 1) - 1.
\]

Moreover, the equality holds only if \( \{u_0, u_{\lfloor W \rfloor - 1}\} = \{v_0, v_t, v_1\} \) for some \( 1 \leq t \leq n \).

Proof. In the left-side of the inequality, the level of each vertex appears twice, except the two ends, for which each appears only once. Hence, the largest possible value is when the two ends are of the smallest levels, which means, \( \{u_0, u_{\lfloor W \rfloor - 1}\} = \{v_0, v_t, v_1\} \) for some \( 1 \leq t \leq n \). The result then follows by some calculation.

Theorem 12 Let \( G = S_{l_1,l_2,l_3,...,l_n} \). Then

\[
\text{rn}(G) \geq \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor + 1.
\]

Moreover, \( f \) is a distance labeling with span equal to this bound if and only if all the following hold: (Note, \( b, c, d \) are only for the case that \( l_1 - l_2 \geq 2 \).)

(a) \( \{u_0, u_{\lfloor W \rfloor - 1}\} = \{v_0, v_t, v_1\} \) for some \( t \).

(b) \( 1 \leq t_j \leq \lfloor W \rfloor - 2 \), for all \( 0 \leq j \leq z \).
(c) \( x_{t_j-1} + x_{t_j} = l_1 - l_2 - (2j + 1) \), for all \( 0 \leq j \leq z \).

(d) For any \( 0 \leq j \leq z \), \( u_{t_j-1} \) and \( u_{t_j+1} \) belong to different legs, unless one of them is the center.

(e) If \( l_1 - l_2 \leq 1 \), then \( x_i = 0 \) for all \( 0 \leq i \leq |V| - 2 \); if \( l_1 - l_2 \geq 2 \), then \( x_i = 0 \) for all \( i \notin \{t_j, t_j - 1 : j = 0, 1, \cdots, z\} \).

**Proof.** Let \( f \) be a distance labeling for \( G = S_{t_1, t_2, \cdots, t_n} \). Consider the case that \( l_1 - l_2 \leq 1 \), or \( l_1 - l_2 \geq 2 \) and \( 1 \leq t_j \leq |V| - 2 \) for any \( v_{i, t_{j-1}} = u_{t_j} \), \( 0 \leq j \leq z \). By the definition of \( x_i \) and Lemmas 10 and 11, we get

\[
\begin{align*}
f(u_{|V|-1}) & = (l_1 + l_2 + 1)(|V| - 1) - 2 \sum_{i=1}^{|V|-2} L(u_i) - L(u_0) - L(u_{|V|-1}) + \sum_{i=0}^{|V|-2} x_i \\
& \geq (l_1 + l_2 + 1) \sum_{i=1}^n l_i - \sum_{i=1}^n l_i(l_i + 1) + 1 + \left[ \frac{l_1 - l_2}{2} \right] \left[ \frac{l_1 - l_2}{2} \right] \\
& = \sum_{i=1}^n l_i(l_1 + l_2 - l_i) + \left[ \frac{l_1 - l_2}{2} \right] \left[ \frac{l_1 - l_2}{2} \right] + 1.
\end{align*}
\]

Moreover, the equality holds if and only if (a, c, d, e) are true.

It remains to show that if (b) fails, then the span of \( f \) is greater than the desired bound. Assume \( t_j = 0 \) for some \( j = 0, 1, 2, \cdots, z \) and \( t' \leq |V| - 2 \) for all \( j' = 0, 1, \cdots, z \). Similar to the proof of Lemma 10, we get

\[
\sum_{i=0}^{|V|-2} x_i \geq \left[ \frac{l_1 - l_2}{2} \right] \left[ \frac{l_1 - l_2}{2} \right] - (l_1 - l_2 - 2j - 1).
\]

Therefore,

\[
\begin{align*}
f(u_{|V|-1}) & = (l_1 + l_2 + 1) \sum_{i=1}^n l_i - 2 \sum_{i=1}^{|V|-2} L(u_i) - L(u_0) - L(u_{|V|-1}) + \sum_{i=0}^{|V|-2} x_i \\
& \geq (l_1 + l_2 + 1) \sum_{i=1}^n l_i - \sum_{i=1}^n l_i(l_i + 1) + l_1 - j + \left[ \frac{l_1 - l_2}{2} \right] \left[ \frac{l_1 - l_2}{2} \right] - (l_1 - l_2 - 2j - 1) \\
& > \sum_{i=1}^n l_i(l_1 + l_2 - l_i) + \left[ \frac{l_1 - l_2}{2} \right] \left[ \frac{l_1 - l_2}{2} \right] + 1.
\end{align*}
\]

Similarly, one can show that if \( t_j = 0 \) and \( t'_j = |V| - 1 \) for some \( 0 \leq j, j' \leq z \), then the span of \( f \) is also greater than the desired bound. \( \blacksquare \)
Depending on the lengths of the legs of a spider, one of the results in Corollary 6 and Theorem 12 provides a lower bound that is better (or not worse) than the other. For instance, Theorem 12 is better when \(2 \leq l_1 - l_2 \leq l_3 + l_4 + \cdots + l_n + 1\); Corollary 6 is better when \((l_1 - l_2)/2 > l_3 + l_4 + \cdots + l_n + 1\); and they both are the same when \(l_1 - l_2 \leq 1\).

4 Spiders Achieving the Bounds

We characterize the spiders whose radio numbers achieve the bound in Theorem 12, and the ones achieving the bound in Corollary 6. We consider them in two separate results.

**Theorem 13** Let \(G = S_{l_1, l_2, \ldots, l_n}\) be a spider. If \(l_1 = l_2\), then

\[
\text{rn}(G) = \begin{cases} 
\sum_{i=1}^{n} l_i(2l_1 - l_i) + 1, & \text{if } n \geq 3 \text{ or } l_1 = 1; \\
\sum_{i=1}^{n} l_i(2l_1 - l_i) + 2, & \text{otherwise}.
\end{cases}
\]

If \(l_1 > l_2\), then

\[
\text{rn}(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor + 1
\]

if and only if \(\sum_{i=2}^{n} l_i \geq \frac{l_1 + l_2 - 1}{2}\).

**Proof.**

**Case \(l_1 = l_2\)** If \(l_1 = 1\), then \(G\) is a star, so the result follows. Assume \(l_1 = l_2 \geq 2\). Suppose the bound in Theorem 12 is achieved by some distance labeling \(f\). By Theorem 12, \(x_i = 0\) for all \(i\), and \(\{u_0, u_{|V|-1}\} = \{v_0, v_t\}\) for some \(t\). This implies that any pair of consecutive vertices belong to different legs, unless one of them is the center. Let \(v_{1, t_1} = u_q\) and \(v_{2, t_2} = u_{q'}\) for some \(1 \leq q, q' \leq |V| - 2\).

Assume \(n = 2\). Consider the three consecutive vertices, \(\{u_{q-1}, u_q, u_{q+1}\}\). Because \(n = 2\), so \(u_{q-1}, u_{q+1} \in V_k\) for some \(k\). Since \(x_i = 0\) for all \(i\), by
Lemma 7 (2), \(\min\{L(u_{q-1}), L(u_{q+1})\} = 0\), implying \(u_{q-1} = v_{0,0}\). The same argument for \(v_{2,t_i} = u_{q'}\) implies that \(u_{q'-1} = v_{0,0}\), a contradiction, as \(q \neq q'\). Therefore, \(rn(G) \geq \sum_{i=1}^{n_l} l_i (2l_1 - l_i) + 2\).

It remains to give distance labelings for \(n = 2\) and \(n \geq 3\), respectively, with the corresponding spans. For both cases, we give a labeling \(f\) with \(f(u_0) = 0\), \(x_i = 0\) for all \(i\), and \(u_{|V|-1} = v_{n,1}\). In addition, for the case \(n \geq 3\), we let \(u_0 = v_{0,0}\); while for the case \(n = 2\), we let \(u_0 = v_{1,1}\). With these properties, it is easy to see that the span of such a labeling \(f\) is equal to the desired bound, respectively.

Throughout the proof, we use a diagram (see below) to describe each labeling \(f\). We first fix the ordering of the vertices \(V = U(f) = \{u_0, u_1, \ldots, u_{|V|-1}\}\). Note that if \(p > l_i\), then \(v_{i,p}\) does not exist. So, for all the diagrams given, when encountering such a “vertex” \(v_{i,p}\), we simply skip it and move on to the next available vertex. Secondly, we put a sign \(\rightarrow\) between two consecutive vertices \(u_i\) and \(u_{i+1}\) to indicate that \(x_i = \ell\). In the case \(x_i = 0\), we just put the sign \(\rightarrow\) between \(u_i\) and \(u_{i+1}\).

For \(l_1 = l_2\) and \(n \geq 3\), \(f\) is defined by: (See Figure 2 for an example.)

\[
\begin{align*}
v_{0,0} & \rightarrow v_{1,1} & \rightarrow v_{2,1} & \rightarrow v_{3,1} & \rightarrow v_{4,1} & \cdots & \rightarrow v_{n,1} \\
& \rightarrow v_{1,1-1} & \rightarrow v_{2,2} & \rightarrow v_{3,1-1} & \rightarrow v_{4,1-1} & \cdots & \rightarrow v_{n,1-1} \\
& \rightarrow v_{1,1-2} & \rightarrow v_{2,3} & \rightarrow v_{3,1-2} & \rightarrow v_{4,1-2} & \cdots & \rightarrow v_{n,1-2} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& \rightarrow v_{1,1} & \rightarrow v_{2,1} & \rightarrow v_{3,1} & \rightarrow v_{4,1} & \cdots & \rightarrow v_{n,1}.
\end{align*}
\]

To verify that \(f\) is a distance labeling, it suffices to show that \(f\) satisfies Lemma 7 (1)(2). Because \(2L(u_j) < 2l_1 + 1 = l_1 + l_2 + 1\) for any \(0 \leq i \leq |V| - 1\), so (1) holds. To show (2), consider a set of consecutive vertices \(\{u_i, u_{i+1}, \ldots, u_j\}\) with \(u_i, u_j \in V_k\) for some \(k\). By the definition of \(f\) in the above, if \(k \geq 3\), then \(j - i \geq 3\), and there exists some \(i+1 \leq q \leq j-1\), such that \(2 (L(u_j) + \min\{L(u_i), L(u_j)\}) \leq l_1 + l_2 + 2\). Combining this with the fact that \(2L(u_t) < l_1 + l_2 + 1\) for every \(t\), (2) is true. If \(k \leq 2\), then there exists some \(i+1 \leq q \leq j-1\) such that \(2 (L(u_j) + \min\{L(u_i), L(u_j)\}) \leq l_1 + l_2 + 1\), so (2) holds.
For $l_1 = l_2$ and $n = 2$ (that is, $G$ is an odd path which was determined in [17], we include it here for completeness), $f$ is defined by:

\[
\begin{align*}
v_{1,1} &\rightarrow v_{2,l_1-1} \rightarrow v_{1,l_1} \rightarrow v_{0,0} \rightarrow v_{2,l_1} \\
\rightarrow v_{1,2} &\rightarrow v_{2,l_1-2} \\
\rightarrow v_{1,3} &\rightarrow v_{2,l_1-3} \\
\vdots &\vdots \\
\rightarrow v_{1,l_1-1} &\rightarrow v_{2,1}.
\end{align*}
\]

Similar to the above, it is easy to show that (1) and (2) in Lemma 7 hold.

**Case $l_1 - l_2 = 1$** (Note, this case includes all even paths.) It suffices to give a distance labeling $f$ which achieves the bound. The labeling $f$ is defined by: (See Figure 2 for an example.)

\[
\begin{align*}
v_{0,0} &\rightarrow v_{1,t_1} \rightarrow v_{2,1} \rightarrow v_{3,t_2} \rightarrow v_{4,t_2} \rightarrow \cdots \rightarrow v_{n,t_2} \\
\rightarrow v_{1,t_1-1} &\rightarrow v_{2,2} \rightarrow v_{3,t_2-1} \rightarrow v_{4,t_2-1} \rightarrow \cdots \rightarrow v_{n,t_2-1} \\
\rightarrow v_{1,t_1-2} &\rightarrow v_{2,3} \rightarrow v_{3,t_2-2} \rightarrow v_{4,t_2-2} \rightarrow \cdots \rightarrow v_{n,t_2-2} \\
\vdots &\vdots \vdots \vdots \vdots \vdots \\
\rightarrow v_{1,2} &\rightarrow v_{2,t_2} \rightarrow v_{3,1} \rightarrow v_{4,1} \rightarrow \cdots \rightarrow v_{n,1} \\
\rightarrow v_{1,1}.
\end{align*}
\]

For any vertex $u_i$, we have $2L(u_i) \leq 2l_1 = l_1 + l_2 + 1$. By a similar argument to the previous cases, one can show that $f$ is a distance labeling.

**Case $l_1 - l_2 \geq 2$** This case implies that $n \geq 3$. First, we prove that if the bound of the radio number is achieved, then \(\sum_{i=3}^{n} l_i \geq \frac{l_1 - l_2 - 1}{2}\). Note that it is trivial if $l_1 - l_2 \leq 3$, as $l_3 \geq 1$.

Assume $l_1 - l_2 \geq 4$. Let $f$ be a distance labeling for $G$ with span equal to the desired bound. We adopt the same notations used in the proof of Theorem 12, let $z = \lfloor \frac{l_1 - l_2 - 2}{2} \rfloor$, and let $v_{1,t_1-j} = u_j$ for $j = 0, 1, \cdots, z$. Then, (a) - (e) in Theorem 12 hold.

**Claim.** For any $\lfloor \frac{l_1 - l_2 + 1}{2} \rfloor \leq i \leq \lceil \frac{l_1 + l_2 + 1}{2} \rceil$, $v_{1,i}$ is not consecutive to any vertex in $V_1 - \{v_{0,0}\}$.

**Proof.** Let $v_{1,i} = u_q$ for some $q$. By Theorem 12 (a), $1 \leq q \leq |V| - 2$, as $i \geq \lceil \frac{l_1 - l_2 + 1}{2} \rceil \geq 2$. By Lemma 7 (2) and Theorem 12 (e), it is enough to show
that $q \neq t_j - 1$, for all $j = 0, 1, 2, \cdots, z$. Suppose to the contrary, $q = t_j - 1$ for some $0 \leq j \leq z$. Since $l_1 - z \geq i$, by Lemma 7 (2), $x_{t_j - 1} \geq 2L(u_q) = 2i \geq l_1 - l_2$, contradicting Theorem 12 (c).

Theorem 12 (d) implies that $u_t \in V_1 - \{v_0, 0\}$ or $u_t \notin V_1 - \{v_0, 0\}$ for $j = 0, 1, 2, \cdots, z$, so $|V(G) - V_1| \geq z$. Combining this with the Claim above, we conclude that $|V(G) - V_1| \geq z + l_2 + 1 \geq \frac{l_1 + l_2 - 1}{2}$.

Now, it remains to give a distance labeling with span achieving the bound. We consider cases separately. It is a routine to check that each of the labelings given is a distance labeling (by Lemma 7) and achieves the bound (by Theorem 12). We sketch the proof for the first one, and leave the details of others to the reader.

Figure 2: Optimal distance labelings for $S_{3,3,3,3,3,3,3}$ and $S_{4,3,3,2,2}$. 
If $l_1 - l_2 = 2$, $f$ is defined by: (See Figure 3 as an example.)

\[
\begin{align*}
v_{0,0} & \to v_{1,t_1} \quad 1 \quad v_{2,1} \\
& \to v_{1,t_1-1} \quad \to v_{3,1} \quad \to v_{4,1} \quad \to v_{5,1} \quad \cdots \quad \to v_{n,1} \\
& \to v_{1,t_1-2} \quad \to v_{2,2} \quad \to v_{3,2} \quad \to v_{4,2} \quad \to v_{5,2} \quad \cdots \quad \to v_{n,2} \\
& \to v_{1,t_1-3} \quad \to v_{2,3} \quad \to v_{3,3} \quad \to v_{4,3} \quad \to v_{5,3} \quad \cdots \quad \to v_{n,3} \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
& \to v_{1,2} \quad \to v_{2,t_2} \quad \to v_{3,t_2} \quad \to v_{4,t_2} \quad \to v_{5,t_2} \quad \cdots \quad \to v_{n,t_2} \\
& \to v_{1,1}.
\end{align*}
\]

To show that $f$ is a distance labeling, observe that $2L(u_q) = l_1 + l_2 + 1$ for any $u_q \neq v_{1,t_1}$, and $2L(v_{1,t_1}) = 2l_1 = (l_1 + l_2 + 1) + 1$. Because $x_1 = 1$, one can show that Lemma 7 (1) holds. To show Lemma 7 (2), consider any set of consecutive vertices $\{u_1, u_{i+1}, \cdots, u_j\}$ with $u_i, u_j \in V_k$ for some $k$. If $i \geq 2$, then there exists some $i + 1 \leq q \leq j - 1$ such that

\[
2\left(L(u_q) + \min\{L(u_i), L(u_j)\}\right) \leq 2l_1 - 2 < l_1 + l_2 + 1,
\]

so (2) holds. If $i = 0$ or $i = 1$, one can also show that (2) holds.

It is easy to check that $f$ satisfies (a) - (e) in Theorem 12, so the span of $f$ is equal to the desired bound.

If $l_1 - l_2 \geq 3$, we consider two sub-cases. Let $A$ be the set of vertices, $A = V(G) - (V_1 \cup V_2)$. We line up the vertices in $A$ by:

\[
A = \{v_{3,1}, v_{4,1}, \cdots, v_{n,1}, v_{3,2}, v_{4,2}, \cdots, v_{n,2}, \cdots, v_{n,t_n}\}.
\]

By assumption and by considering the parity of $l_1 - l_2$, we get $|A| \geq z + 1$. Let $A[z+1]$ be the set of the first $z + 1$ vertices in $A$, and let

\[
A_i = \{v : L(v) = i\} \cap (A - A[z+1])
\]

We denote $A(z+1)$ the first unlabeled yet vertex in the line up of $A[z+1]$. When we encounter $A_i$ in the diagrams below, we color all the vertices in $A_i$ one by one (in any order) with $x_q = 0$ if $u_{q-1}, u_q \in A_i$. If $A_i = \emptyset$, we skip it.

For both sub-cases, it is a routine to check Lemma 7 and Theorem 12. We shall leave the details to the reader. See Figure 4 for examples.
Figure 3: An optimal distance labeling for $S_{5,3,2,2,1}$.

**Sub-case: $l_1 - l_2$ is even.** Then $z = \frac{l_1 - l_2 - 2}{2}$. The labeling is defined as:

$$
\begin{align*}
&v_{0,0} \rightarrow v_{1,l_1 - z} \quad \xrightarrow{1} A(z + 1) \\
&\quad \rightarrow v_{1,l_1 - z - 1} \rightarrow v_{2,1} \quad \rightarrow A_{l_2} \\
&\quad \rightarrow v_{1,l_1 - z - 2} \rightarrow v_{2,2} \quad \rightarrow A_{l_2 - 1} \\
&\quad \rightarrow v_{1,l_1 - z - 3} \rightarrow v_{2,3} \quad \rightarrow A_{l_2 - 2} \\
&\vdots \quad \vdots \quad \vdots \\
&\rightarrow v_{1,z+2} \rightarrow v_{2,l_2} \rightarrow A_1 \\
&\rightarrow v_{1,z+1} \rightarrow A(z + 1) \\
&\rightarrow v_{1,l_1} \xrightarrow{2z+1} v_{1,z} \quad \rightarrow A(z + 1) \\
&\rightarrow v_{1,l_1-1} \xrightarrow{2z-1} v_{1,z-1} \quad \rightarrow A(z + 1) \\
&\rightarrow v_{1,l_1-2} \xrightarrow{2z-3} v_{1,z-2} \quad \rightarrow A(z + 1) \\
&\vdots \quad \vdots \quad \vdots \\
&\rightarrow v_{1,l_1 - z+2} \xrightarrow{5} v_{1,2} \quad \rightarrow A(z + 1) \\
&\rightarrow v_{1,l_1 - z+1} \xrightarrow{3} v_{1,1}.
\end{align*}
$$

Note that the very first $A(z + 1)$ appeared in the above is $v_{3,1}$ (with level 1), and any other $A(z + 1)$ is a vertex with level at most $l_2$. 

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Figure 4: Optimal distance labelings for $S_{7,3,3,1}$ and $S_{7,2,1,1,1,1}$. 
Sub-case: $l_1 - l_3$ is odd. Then $z = \frac{l_1 - l_3 - 3}{2}$. The labeling is defined by:

\[
\begin{align*}
v_{0,0} & \rightarrow v_{1,l_1 - z - 1} \rightarrow v_{2,1} \rightarrow A_{l_2} \rightarrow A_{l_2 - 1} \rightarrow A_{l_2 - 2} \\
v_{1,l_1 - z - 2} & \rightarrow v_{2,2} \\
v_{1,l_1 - z - 3} & \rightarrow v_{2,3} \\
\vdots & \vdots \\
v_{1,z+3} & \rightarrow v_{2,l_2} \rightarrow A_{1} \\
v_{1,z+2} & \rightarrow A(z + 1) \\
v_{1,l_1} & \rightarrow 2v_{1,z} \rightarrow A(z + 1) \\
v_{1,l_1 - 1} & \rightarrow 2v_{1,z - 1} \rightarrow A(z + 1) \\
\vdots & \vdots \\
v_{1,l_1 - (z-1)} & \rightarrow 2v_{1,2} \rightarrow A(z + 1) \\
v_{1,l_1 - z} & \rightarrow 2v_{1,1}.
\end{align*}
\]

In the following, we characterize the spiders achieving the bound in Corollary 6. Note that, if $l_1 - l_2 \leq 1$ or $l_3 = 0$, then the radio number has been determined in Theorem 13. Hence, we assume $l_1 - l_2 \geq 2$ and $l_3 \geq 1$.

**Theorem 14** Let $G = S_{l_1,l_2,\ldots,l_n}$ be a spider with $l_1 - l_2 \geq 2$ and $l_3 \geq 1$. Then

\[
\text{rn}(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + 1 + 2\left[\frac{l_1 - (l_2 + \cdots + l_n - 1)}{2}\right] + 2\left[\frac{l_1 - (l_2 + \cdots + l_n + 1)}{2}\right],
\]

if and only if $l_3 = 1$, $n = 3$, and $l_1 - l_2 \geq 3$ is odd. (Note, the last term in the above becomes 0 if $l_1 \leq l_2 + l_3 + \cdots + l_n + 1$.)

**Proof.** Assume that the equality holds. Let $f$ be an optimal distance labeling with ordering $U(F) = (u_0, u_1, u_2, \ldots, u_{m-1})$, where $m = l_1 + l_2 + \cdots + l_n + 1$. If $l_1 \leq m/2$, then by assumption, $\text{rn}(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + 1$. However, by Theorem 13, $\text{rn}(G) \geq \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + 2$ (as $l_1 - l_2 \geq 2$), a contradiction.

Hence, $l_1 > m/2$. By Lemma 1, we assume, by symmetry, that a weight center is $w^* = v_{1,k} = u_0$, where $k = l_1 - \lfloor m/2 \rfloor$. There are exactly two
branches for \( w^* \). By Theorems 4, 5, and Corollary 6, \( f \) satisfies (1 - 3) in Theorem 4.

**Claim.** If \( u_q = v_{1,l_1} \), then \( 1 \leq q \leq m - 2 \), and

\[
\min\{d(u_{q-1}, u_q), d(u_q, u_{q+1})\} \leq (l_1 + l_2 + 1)/2.
\]

**Proof** Assume \( u_q = v_{1,l_1} \). By Theorem 4 (2), \( 1 \leq q \leq m - 2 \) (since \( l_1 \geq 2 \)). Suppose to the contrary that \( \min\{d(u_{q-1}, u_q), d(u_q, u_{q+1})\} > (l_1 + l_2 + 1)/2 \).

Assume \( u_{q-1} \) and \( u_{q+1} \) are on the same leg. Assume \( d(u_{q-1}, u_q) \geq d(u_q, u_{q+1}) \) (the other case is similar). By considering the path containing \( u_{q-1}, u_q, u_{q+1} \), we have \( d(u_{q-1}, u_{q+1}) = d(u_{q-1}, u_q) - d(u_q, u_{q+1}) \). By definition, \( f(u_{q+1}) - f(u_{q-1}) \geq d+1-d(u_{q-1}, u_{q+1}) = d+1-d(u_{q-1}, u_q) + d(u_q, u_{q+1}) \). By Theorem 4 (3), \( f(u_{q+1}) - f(u_{q-1}) = 2(d+1) - d(u_{q-1}, u_q) - d(u_q, u_{q+1}) \). Combining these together, we have \( l_1 + l_2 + 1 = d + 1 \geq 2d(u_q, u_{q+1}), \) contradicting that \( d(u_q, u_{q+1}) > (l_1 + l_2 + 1)/2 \).

Assume \( u_{q-1} \) and \( u_{q+1} \) belong to different legs. Then \( d(u_{q-1}, u_q) + d(u_q, u_{q+1}) - d(u_{q-1}, u_{q+1}) \geq 2l_1 \). Similar to the above, we have \( 2(d+1) - d(u_{q-1}, u_q) - d(u_q, u_{q+1}) \geq d + 1 - d(u_{q-1}, u_{q+1}) \). Hence, \( l_1 + l_2 + 1 = d + 1 \geq 2l_1 \), contradicting the hypothesis \( l_1 - l_2 \geq 2 \).

Assume \( l_3 + l_4 + \cdots + l_n \geq 2 \). Then \( d(v_{1,l_1}, v_{1,k}) = l_1 - k \geq (l_1 + l_2 + 2)/2 \). This implies that \( d(v_{1,l_1}, v) > (l_1 + l_2 + 1)/2 \), for any vertex \( v \) on the branch opposite to \( v_{1,l_1} \), contradicting Theorem 4 (1) and the Claim. Therefore, \( l_3 + \cdots + l_n \leq 1 \). By the assumption that \( l_3 \geq 1 \), we have \( l_3 = 1 \) and \( n = 3 \).

Assume \( l_3 = 1, n = 3, \) and \( l_1 - l_2 \) is even. Then \( m = l_1 + l_2 + 2 \) is even, and \( l_1 - k = m/2 \). So the distance between \( v_{1,l_1} \) and any other vertex in the opposite branch is at least \( m/2 > (l_1 + l_2 + 1)/2 \), contradicting the Claim.

It remains to give a distance labeling \( f \) satisfying (1 - 3) in Theorem 4, for \( l_3 = 1, n = 3, \) and \( l_1 - l_2 \geq 3 \) is odd. Note, \( m = l_1 + l_2 + 2 \) is odd. By Lemma 1, \( w^* = v_{1,k} \) is the weight center, where \( k = (l_1 - l_2 - 1)/2 \). Define the ordering \( U(f) \) by the following three steps:

1) \( u_0 = v_{1,k}, \ u_1 = v_{1,l_1}, \ u_2 = v_{1,k-2} \) (or \( u_2 = v_{2,1} \) if \( k = 1 \));
2) move back and forth on the path $V_0 \cup V_1$, about the weight center $v_{1,k}$, with distances alternating between $(l_1 + l_2 + 1)/2$ and $(l_1 + l_2 + 3)/2$, until we reach the vertex $v_{1,k+1}$. That is, $u_3$ is a vertex with distance $(l_1 + l_2 + 1)/2$ from $u_2$ (indeed $u_3 = v_{1,l_1-2}$), and $u_4$ has distance $(l_1 + l_2 + 3)/2$ away from $u_3$, etc;

3) $u_{m-3} = v_{3,1}, u_{m-2} = v_{1,l_1-1}$ and $u_{m-1} = v_{1,k-1}$.

For all $i$, we let $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$. It is straightforward to check that $f$ is a distance labeling satisfying (1 - 3) in Theorem 4. We leave the details to the reader.

References


