An introduction to expander families and Ramanujan graphs

Tony Shaheen
CSU Los Angeles
Before we get started on expander graphs I want to give a definition that we will use in this talk.

A graph is **regular** if every vertex has the same degree (the number of edges at that vertex).

A 3-regular graph.
Now for the motivation behind expander families...

Think of a graph as a communications network.
Two vertices can communicate directly with one another iff they are connected by an edge.
Communication is instantaneous across edges, but there may be delays at vertices.
Edges are expensive.
Our goal: Let $d$ be a fixed integer with $d > 1$.

Create an infinite sequence of $d$-regular graphs

$$X_1, X_2, X_3, X_4, X_5, \ldots$$

where

1. the graphs are getting bigger and bigger (the number of vertices of $X_n$ goes to infinity as $n$ goes to infinity)

2. each $X_n$ is as good a communications network as possible.
Questions:

1. How do we measure if a graph is a good communications network?

2. Once we have a measurement, can we find graphs that are optimal with respect to the measurement?
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1. How do we measure if a graph is a good communications network?

2. Once we have a measurement, how good can we make our networks?

Let’s start with the first question.
Consider the following graph:
Let’s look at the set of vertices that we can reach after \( n \) steps, starting at the top vertex.
Here is where we can get to after 1 step.
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We would like to have many edges going outward from there.
Here is where we can get to after 2 steps.
For any set $S$ of vertices, we would like to have many edges from $S$ to its complement.
The set of edges from \( S \) to its complement is called the *boundary* of \( S \), denoted \( \partial S \).
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**Example:**

$$|S| = 5$$
$$|\partial S| = 6$$
Let $G$ be a graph. Define $h(G)$ to be the minimum value of

$$\frac{|\partial S|}{|S|}$$

over all sets $S$ containing no more than half the vertices.
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over all sets $S$ containing no more than half the vertices.

$h(G)$ is called the expansion constant of $G$. 
Here’s what the expansion constant tells us.

Even in a worst case, for any small set $S$, we have at least $h(G)|S|$ edges going from $S$ to its complement.
Take-home Message #1:

The expansion constant is one measure of how good a graph is as a communications network.
We want $h(X)$ to be **BIG!**
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If a graph has small degree but many vertices, this is not easy.
Consider the cycles graphs:

$C_3$  $C_4$  $C_5$  $C_6$  ...
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\[ C_3 \quad C_4 \quad C_5 \quad C_6 \] …

Each is 2-regular.
The number of vertices goes to infinity.
Let $S$ be the bottom half.
\[ h(C_n) \leq \frac{|\partial S|}{|S|} = \frac{2}{\binom{n}{2}} = \frac{4}{n} \rightarrow 0 \]
We say that a sequence of regular graphs is an **expander family** if

- All the graphs have the same degree
- The number of vertices goes to infinity
- There exists a positive lower bound $r$ such that the expansion constant is always at least $r$. 
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Existence: Pinsker 1973
First explicit construction: Margulis 1973
So far we have looked at the combinatorial way of looking at expander families.

Let’s now look at it from an algebraic viewpoint.
We form the *adjacency matrix* of a graph as follows:

\[ A = \begin{pmatrix}
  0 & 0 & 0 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 1 & 1 \\
  1 & 1 & 1 & 0 & 0 & 0 \\
  1 & 1 & 1 & 0 & 0 & 0 \\
  1 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix} \]
Facts about the eigenvalues of a $d$-regular connected graph $G$ with $n$ vertices:
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• The eigenvalues satisfy

\[-d \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \ldots \leq \lambda_1 < \lambda_0 = d\]
Facts about the eigenvalues of a $d$-regular connected graph $G$ with $n$ vertices:

- They are all real.
- The eigenvalues satisfy
  \[-d \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \ldots \leq \lambda_1 < \lambda_0 = d\]
- The second largest eigenvalue $\lambda_1$ satisfies
  \[
  \frac{d - \lambda_1}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_1)}
  \]

(Alon-Dodziuk-Milman-Tanner)
Notice that the quantity $d - \lambda_1$ appears on both sides.

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Notice that the quantity $d - \lambda_1$ appears on both sides.

Recall that $d$ is fixed.

\[ \frac{d - \lambda_1}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_1)} \]

(Alon-Dodziuk-Milman-Tanner)
Notice that the quantity $d - \lambda_1$ appears on both sides.

Recall that $d$ is fixed.

The inequality below tells us that $h(G)$ is big iff $\lambda_1$ is small.

$$\frac{d - \lambda_1}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_1)}$$

(Alon-Dodziuk-Milman-Tanner)
Take-home Message #2:

The expansion constant is big iff $\lambda_1$ is small.
For any $d$-regular graph $G$, its second largest eigenvalue $\lambda_1$ must lie above the red curve.
The red curve has a horizontal asymptote at $2\sqrt{d - 1}$.
In other words, $2\sqrt{d} - 1$ is asymptotically the smallest that $\lambda_1$ can be.
We say that a $d$-regular graph $X$ is **Ramanujan** if all the non-trivial eigenvalues $\lambda$ of $X$ (the ones that aren’t equal to $d$ or -$d$) satisfy

$$|\lambda| \leq 2\sqrt{d - 1}$$
Hence, if $X$ is Ramanujan then

$$\lambda_1 \leq 2\sqrt{d - 1}$$
Take-home Message #3:

Ramanujan graphs essentially have the smallest possible $\lambda_1$. 

A family of d-regular Ramanujan graphs is an expander family.
KNOWN:
There exists a family of $d$-regular Ramanujan graphs if $d - 1$ is a prime power.

(Lubotzky-Phillips-Sarnak 1988 for primes, Morgenstern 1994 for prime powers)
OPEN PROBLEM:
Let $d \geq 3$ such that $d - 1$ is not a prime power. ($d = 7$ is the smallest such $d$.)
Does there exist a family of $d$-regular Ramanujan graphs?
SHAMELESS SELF-PROMOTION!!!

Expander families and Cayley graphs –
A beginner’s guide

by Mike Krebs and Anthony Shaheen