Wikipedia states that, “The first spiral monohedral tiling was discovered by Heinz Voderberg in 1936, with the Voderberg tiling having a unit tile that is a nonconvex enneagon.” Although we don’t dispute this, we wonder if a simple triangle wouldn’t have been a more obvious choice. Today, we realize that there are an infinite number of spiral monohedral tilings. So, other than its striking good looks, what distinguishes the Voderberg tile and captures the imagination? Grünbaum and Shephard (1986) nail it: “Voderberg discovered that these tiles have the remarkable property that two tiles can completely surround a third tile of the same shape, or even two other tiles.” Figure 1 shows the enclosure properties of the Voderberg tile.

The problem is that we are left with the overriding feeling that this is unique to this enneagon with a 12º vertex. But it isn’t. In fact, there are an infinite number of such tiles; and while they derive from the Voderberg, it has nothing to do with having either nine sides or a specific vertex angle. Rather, it has to do with the three red lines depicted in Figure 2, as we shall see.

In this paper we define a new class of tiles for which two tiles can completely enclose one or two more copies of the same tile. There are no limitations as to the number of sides or the vertex angle, although only certain angles will admit radial and spiral tiling.

We propose that the three red lines of unit length in Figure 2 are sufficient to ensure this enclosure. The tile is completed by appropriately connecting A’-A and B’-B, subject to three simple rules: 1. the lines are congruent; 2. the lines are antisymmetric; and 3. the lines do not intersect. Clearly, O-A’-A is equal to O-B’-B rotated by $e^{i\alpha}$. All such constructs are guaranteed to meet the enclosure property criteria. Moreover, for integer values of $\pi/\alpha$, aperiodic radial and spiral tiling will ensue with normal triangle substitution tiling [see, for example, Waldman (2014)]. For arbitrary values of $\alpha$, periodic tiling is always available, so it is not without merit.

The gallery of images following Figure 2 demonstrates the enclosure and tiling properties of randomly generated tiles, except for the continuous curves, which are composed of Cornu spiral segments. The spiral tilings were generated with a Matlab program that is freely available from Waldman (2014). The associated animations show (1) a double Cornu tile enclosure with variable vertex angle and (2) a 3º-vertex double Cornu spiral with a 1000-fold zoom-in on the central fan; over 60 tiles tessellate in perfect mathematical precision at the origin.

Finally, we acknowledge that this work was inspired by a tiling in Glassner (1999), which in turn was inspired by the Voderberg spiral. We had never seen a Voderberg-type spiral with other than a 12º vertex. We first duplicated Glassner’s work, albeit mathematically, rather than graphically, and then took off from there.

We welcome your comments.
Figure 1: The Voderberg tile enclosure property.

Figure 2: Three (red) lines that are sufficient to ensure enclosure.

A gallery of images of the new tiles and tilings follows.
Figure 3: Example of random vertex angles: random 13-gon (above) and 21-gon (below).
Figure 4: 22.5° random 9-gon: enclosure (top), spiral (bottom).
Figure 5: $6^\circ$ random 21-gon: enclosure (top), spiral (bottom).
Figure 6: Example of random vertex angles: single Cornu (above), double Cornu (below).
Figure 7: 20° single Cornu: enclosure (top), spiral (bottom).
Figure 8: 6° double Cornu: enclosure (top), spiral (bottom)
References


